CSE 107 10-7-22

- hw 2 : problem 9 deleted
  turn in 1-8 only

- today : finish 2.4
  Expectation, mean, variance

Ex. (previous)

Suppose X has PMF

$$P_X(x) = \begin{cases} .2 & \text{if } x = -1 \\ .3 & \text{if } x = 1 \\ .5 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

previously computed

$E[X] = 1.1$ (1st moment)

$E[X^2] = 2.5$ (2nd moment)

now

$$Var(X) = E[(X - E[X])^2]$$

$$= \sum_x (x-(1.1))^2 p_X(x)$$

$$= (-1-1.1)^2(.2) + (1-1.1)^2(.3) + (2-1.1)^2(.5)$$

$$= \cdots = \boxed{1.29}$$

and

$$\sigma_X = \sqrt{1.29} = \boxed{1.1358}$$

simpler formula for variance

Variance in terms of moments

$$Var(X) = E[X^2] - (E[X])^2$$

Proof:

$$\mathrm{Var}(X) = E[(X - E[X])^2]$$

$$= \sum_x (x - E[X])^2 \cdot p_X(x)$$

$$= \sum_x (x^2 - 2E[X]x + E[X]^2) p_X(x)$$

$$= \sum_x \left(x^2 p_X(x) - 2E[X] x p_X(x) + E[X]^2 \cdot p_X(x)\right)$$

$$= \sum_x x^2 p_X(x) - 2E[X] \cdot \sum_x x p_X(x) + E[X]^2 \cdot \sum_x p_X(x)$$

$$= E[X^2] - 2E[X] \cdot E[X] + E[X]^2$$

$$= E[X^2] - E[X]^2$$

∎

Ex (Previous)

$$Var(X) = E[X^2] - E[X]^2$$

$$= (2.5) - (1.1)^2$$

$$= \boxed{1.29}$$

Ex. let $Y = aX + b$, then recall

$$E[Y] = E[aX+b] = aE[X] + b$$

lets find 2^{nd} moment:

$$E[Y^2] = E[(aX+b)^2]$$

$$= \sum_x (ax+b)^2 p_X(x)$$

$$= \sum_x (a^2x^2 + 2abx + b^2) p_X(x)$$

$$= a^2 \sum_x x^2 p_X(x) + 2ab \sum_x x p_X(x) + b^2 \sum_x p_X(x)$$

$$= a^2 E[X^2] + 2abE[X] + b^2$$

now find $Var(Y)$:

$$Var(Y) = Var(aX + b)$$

$$= E[(aX+b)^2] - E[aX+b]^2$$

$$= a^2 E[X^2] + 2abE[X] + b^2$$

$$- (aE[X] + b)^2$$

$$= a^2 E[X^2] + \cancel{2abE[X]} + \cancel{b^2}$$

$$- (a^2 E[X]^2 + \cancel{2abE[X]} + \cancel{b^2})$$

$$= a^2 (E[X^2] - E[X]^2)$$

$$= a^2 Var(X)$$

Summary:

- $E[aX+b] = aE[X]+b$
- $Var(aX+b) = a^2 Var(X)$

<u>mean & Variance of common r.v.s</u>

<u>Ex</u>. Bernoulli, Parameter p

$$P_X(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \end{cases}$$

$$E[X] = 1 \cdot p + 0 \cdot (1-p) = p$$

$$E[X^2] = 1^2 \cdot p + 0^2 \cdot (1-p) = p$$

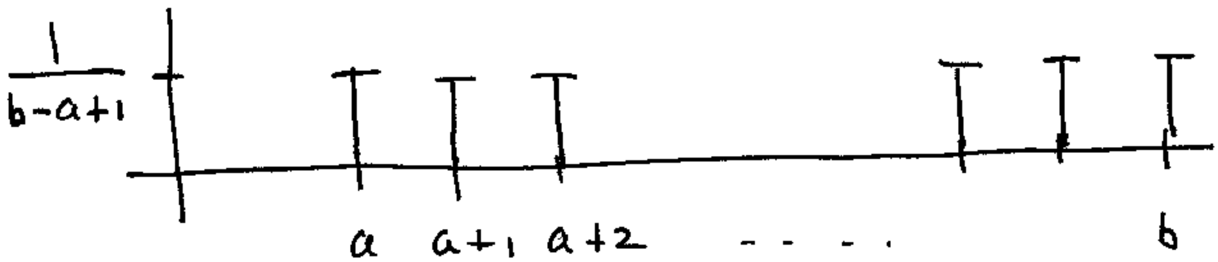
$$Var(X) = E[X^2] - E[X]^2$$

$$= p - p^2 = p(1-p)$$

<u>Ex.</u> Discrete uniform r.v.

Let $a, b \in \mathbb{Z}$ and suppose X distributes its mass uniformly over integers k

$$a \leq k \leq b$$



Thus

$$P_X(k) = \begin{cases} \dfrac{1}{b-a+1} & \text{if } a \leq k \leq b \\ 0 & \text{otherwise} \end{cases}$$

so

$$E[X] = \sum_{k=a}^{b} k \cdot \frac{1}{b-a+1} = \frac{1}{b-a+1} \cdot \sum_{k=a}^{b} k$$

$$= \frac{1}{b-a+1} \left(\sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k \right)$$

Recall: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$

$$\therefore E[X] = \frac{1}{b-a+1}\left(\frac{b(b+1)}{2} - \frac{a(a-1)}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{b-a+1}\right)\left(b^2+b-a^2+a\right)$$

$$= \frac{1}{2}\left(\frac{1}{b-a+1}\right)\left((b^2-a^2)+(b+a)\right)$$

$$= \frac{1}{2}\left(\frac{1}{b-a+1}\right)\left((b-a)(b+a)+(b+a)\right)$$

$$= \left(\frac{b+a}{2}\right)\left(\frac{1}{\cancel{b-a+1}}\right)\left(\cancel{b-a+1}\right)$$

$$= \frac{a+b}{2}$$

To compute $Var(X)$, 1st set

$$Y = X + (1-a)$$

note: $Var(Y) = Var(X+(1-a)) = Var(X)$

let $n = b - a + 1$ and observe Y is uniformily dist. over integers

$$1 \le k \le n$$

since

$$X = a \Rightarrow Y = a + (1-a) = 1$$

$$X = b \Rightarrow Y = b + (1-a) = b - a + 1 = n$$

$$\therefore \quad E[Y] = \frac{n+1}{2}$$

$$E[Y^2] = \sum_{k=1}^{n} k^2 \cdot \frac{1}{n} = \frac{1}{n} \cdot \sum_{k=1}^{n} k^2$$

Recall: $\displaystyle\sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}$

$$\therefore \quad E[Y^2] = \frac{1}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

Therefore

$$Var(X) = Var(Y)$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \ldots \text{ algebra } \ldots$$

$$= \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$$

$$= \frac{(b-a)(b-a+2)}{12}$$

<u>Ex</u> the Poisson r.v., Parameter λ.

$$P_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad (k = 0, 1, 2, \ldots)$$

we'll show that

(1) $E[X] = \lambda$ ✓

(2) $E[X^2] = \lambda^2 + \lambda$ ✓

Proof of (1) :

$$E[X] = \sum_{k=0}^{\infty} k\, e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot \sum_{k=1}^{\infty} k\, \frac{\lambda^k}{k!}$$

$$= \lambda \cdot e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda \cdot e^{-\lambda} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) \quad \left\{ \begin{array}{l} \text{replace} \\ k \leftarrow k+1 \\ k+1=1 \Leftrightarrow k=1 \end{array} \right.$$

$$= \lambda \cdot \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}} = \lambda$$

Proof of (2)

$$E[X^2] = \sum_{k=\cancel{0}\,1}^{\infty} k^2 \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \lambda \cdot e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \cdot \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda \cdot e^{-\lambda} \cdot \sum_{k=0}^{\infty} (k+1) \cdot \frac{\lambda^k}{k!} \quad \left\{ \begin{array}{l} \text{replace} \\ k \leftarrow k+1 \end{array} \right.$$

$$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \left(k \frac{\lambda^k}{k!} + \frac{\lambda^k}{k!} \right)$$

$$= \lambda \left(e^{-\lambda} \cdot \sum_{k=\not{0}\,1}^{\infty} k \frac{\lambda^k}{k!} + e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right)$$

$$= \lambda (\lambda + e^{-\lambda} \cdot e^{\lambda})$$

$$= \lambda (\lambda + 1) = \lambda^2 + \lambda$$

Now we have

$$Var(X) = E[X^2] - E[X]^2$$

$$= (\cancel{\lambda^2} + \lambda) - \cancel{\lambda^2}$$

$$= \lambda$$

Summary:

$$E[X] = \lambda$$

$$Var[X] = \lambda$$