The Poisson random variable

The Poisson r.v. has PMF

$$p_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad (k = 0, 1, 2, \ldots)$$

Parameter : $\lambda > 0$.

First observe

$$\sum_{k=0}^{\infty} p_X(k) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot e^{\lambda} = 1.$$

we have an approximation:

$$e^{-\lambda} \cdot \frac{\lambda^k}{k!} \approx \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \quad (k=0,1,2,\dots)$$

where $\lambda = n \cdot p$, n is large, p is small

more precisely: let $\lambda > 0$ be a constant and $p = \frac{\lambda}{n}$, then

$$\lim_{n\to\infty} \left[\binom{n}{k} \cdot \left(\frac{\lambda}{n}\right)^k \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k} \right] = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

Ex. let $n = 200$, $p = .01$, $\lambda = n \cdot p = 2$

The Prob. of $k = 4$ successes in 200 ind. trials with Prob. of successes .01 is

$$\binom{200}{4}(.01)^4(.99)^{196} = \underbrace{.0902}197$$

This well approximated by

$$e^{-2} \cdot \frac{2^4}{4!} = \underbrace{.0902}235$$

2.3 Functions of a random var.

Given a r.v. X, we can compose it with a fcn.

$$g : \mathbb{R} \to \mathbb{R}$$

to obtain: $Y = g(X) = g \circ X$

$$X : \Omega \to \mathbb{R}$$

so $Y = g \circ X : \Omega \to \mathbb{R}$ is itself a random variable

Ex. Suppose X has PMF

$$P_X(x) = \begin{cases} .2 & \text{if } x=-1 \\ .3 & \text{if } x=1 \\ .5 & \text{if } x=2 \\ 0 & \text{otherwise} \end{cases}$$

let $Y = X^2$ (i.e. $g(x) = x^2$).

Find the PMF of Y.

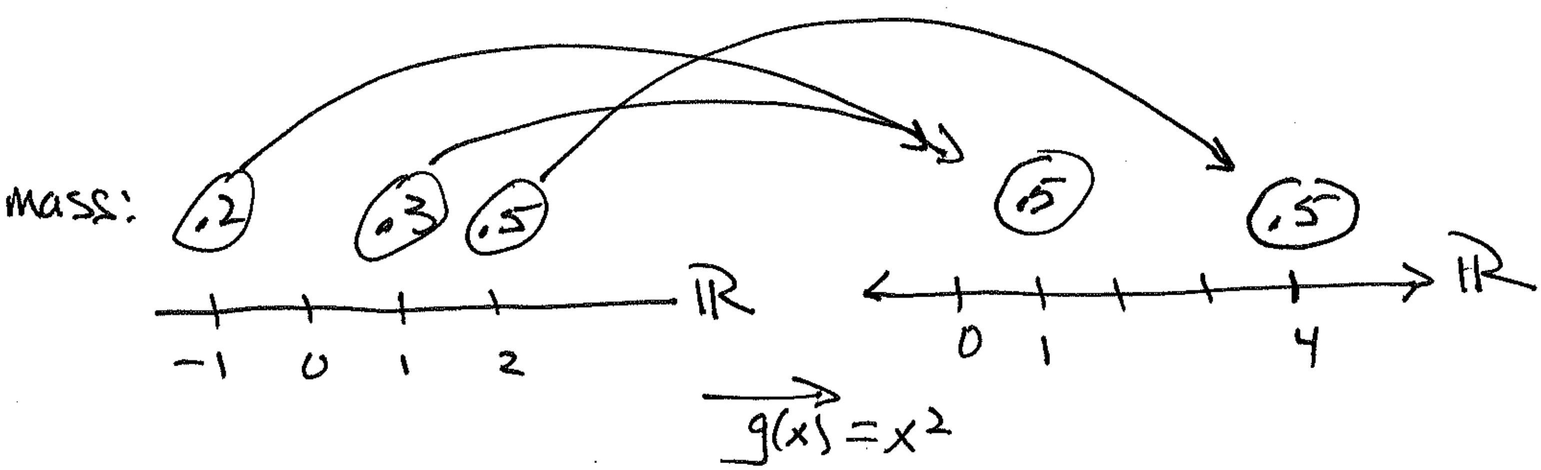
$$\begin{aligned} P_Y(y) &= P(Y=y) \\ &= P(X^2 = y) \\ &= P(X = \pm\sqrt{y}) \\ &= P(X=\sqrt{y} \text{ or } X=-\sqrt{y}) \\ &= P(\{X=\sqrt{y}\} \cup \{X=-\sqrt{y}\}) \end{aligned}$$

$$= P(X = \sqrt{y}) + P(X = -\sqrt{y})$$

$$= \begin{cases} \cancel{.2 \quad \text{if } \sqrt{y} = -1} \\ .3 \quad \sqrt{y} = 1 \\ .5 \quad \sqrt{y} = 2 \\ 0 \quad \text{otherwise} \end{cases} + \begin{cases} .2 \quad \text{if } -\sqrt{y} = -1 \\ \cancel{.3 \quad -\sqrt{y} = 1} \\ \cancel{.5 \quad -\sqrt{y} = 2} \\ 0 \quad \text{otherwise} \end{cases}$$

$$= \begin{cases} .3 & \text{if } y = 1 \\ .5 & y = 4 \\ 0 & \end{cases} + \begin{cases} .2 & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} .5 & \text{if } y = 1 \\ .5 & \text{if } y = 4 \\ 0 & \text{otherwise} \end{cases}$$



2.4 Expectation, mean, Variance

Defn

The expected value of a r.v. X is

$$E[X] = \sum_x x \cdot p_X(x)$$

where $p_X(x)$ is the PMF of X, sum over all $x \in \text{range}(X)$.

$E[X]$ also called: expectation, mean, average.

$E[X]$ is actually the center of 'mass'.

Ex. (previous)

$$E[X] = \sum_x x \cdot p_X(x)$$

$$= (-1)(.2) + 1 \cdot (.3) + 2 \cdot (.5)$$

$$= \boxed{1.1}$$

$$E[X^2] = E[Y]$$

$$= \sum_y y \cdot p_Y(y)$$

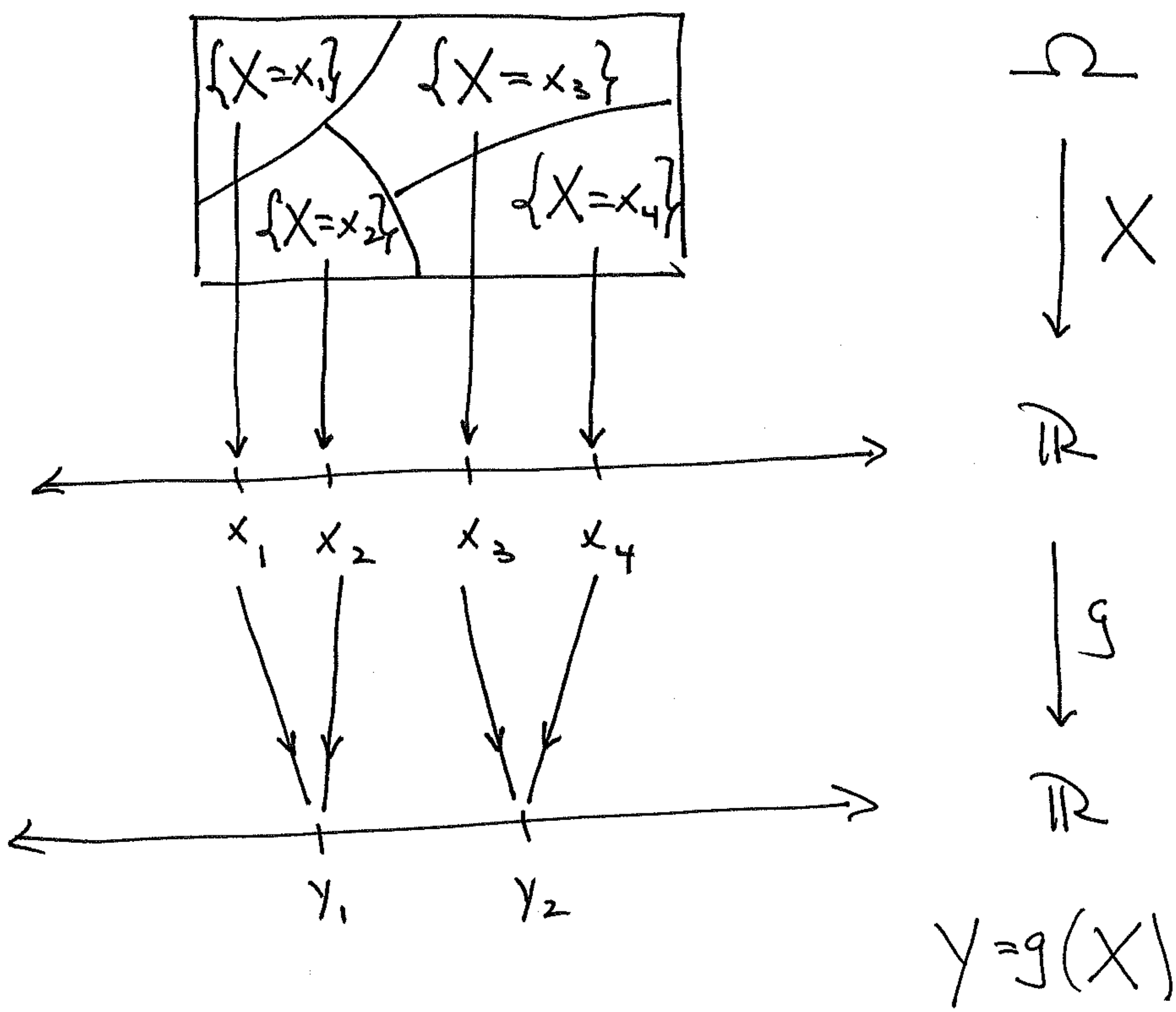
$$= 1 \cdot (.5) + 4 \cdot (.5) = \boxed{2.5}$$

note: $E[X^2] \neq (E[X])^2$

Expected value rule

$$E[g(X)] = \sum_x g(x) \cdot p_X(x)$$

Picture:



$$E[Y] = y_1 \cdot p_Y(y_1) + y_2 \cdot p_Y(y_2)$$

$$= y_1 \left(p_X(x_1) + p_X(x_2) \right) + y_2 \left(p_X(x_3) + p_X(x_4) \right)$$

$$= y_1 p_X(x_1) + y_1 p_X(x_2) + y_2 p_X(x_3) + y_2 p_X(x_4)$$

$$= g(x_1) p_X(x_1) + g(x_2) p_X(x_2) + g(x_3) p_X(x_3) + g(x_4) p_X(x_4)$$

$$= \sum_x g(x) \cdot p_X(x)$$

Proof

1^{st} observe

$$\{Y = y\} = \{\omega \mid Y(\omega) = y\}$$

$$= \{\omega \mid g(X(\omega)) = y\}$$

$$= \bigcup_{\{x \mid g(x) = y\}} \{\omega \mid X(\omega) = x\}$$

$$P_Y(y) = P(Y = y)$$

$$= \sum_{\{x \mid g(x) = y\}} P(X = x)$$

$$= \sum_{\{x \mid g(x) = y\}} P_X(x)$$

so

$$E[Y] = \sum_y y \cdot P_Y(y)$$

$$= \sum_y y \left(\sum_{\{x \mid g(x) = y\}} P_X(x) \right)$$

$$= \sum_y \sum_{\{x \mid g(x) = y\}} y \cdot P_X(x)$$

$$= \sum_y \sum_{\{x \mid g(x) = y\}} g(x) P_X(x)$$

$$= \sum_x g(x) p_X(x)$$

Ex. (Previous again)

$$E[X^2] = (-1)^2 \cdot (.2) + 1^2 \cdot (.3) + 2^2 \cdot (.5)$$

$$= \boxed{2.5}$$

Ex. let $Y = aX + b$ $(a \neq 0)$. Then

$$E[Y] = E[aX + b]$$

$$= \sum_x (ax + b) p_X(x)$$

$$= \sum_x \left(ax p_X(x) + b p_X(x)\right)$$

$$= a \sum_x x p_X(x) + b \sum_x p_X(x)$$

$$= a\,E[X] + b$$

so $\boxed{E[aX+b] = a\,E[X] + b}$

in particular:

- $E[aX] = a\,E[X]$

- $E[b] = b$

Defn

The <u>n^{th} moment of X</u> $(n = 1, 2, \ldots)$

is

$$E[X^n] = \sum_x x^n \cdot p_X(x)$$

Defn

The variance of X is the mean of the r.v. $(X - E[X])^2$

$$Var(X) = E\left[(X - E[X])^2\right]$$

observe, since the r.v. $(X - E[X])^2$ is non-negative, $Var(X) \geq 0$.

Defn

The standard deviation of X is

$$\sigma_X = \sqrt{Var(X)}$$