CSE 107 10-4-22

General definition of independence

We say $\{A_1, A_2, \ldots, A_n\}$ is independent

iff

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdots P(A_{i_k})$$

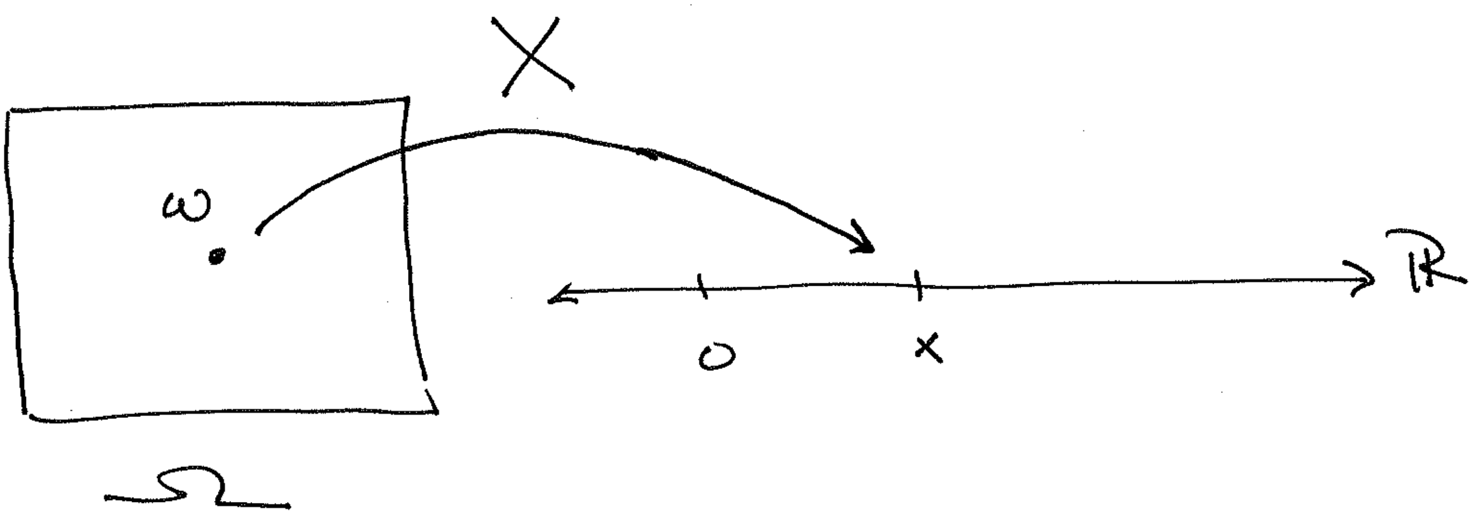
for every $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$

2.1 Random Variables

Defn

A random variable is a function

$$X : \Omega \longrightarrow \mathbb{R}$$



notation

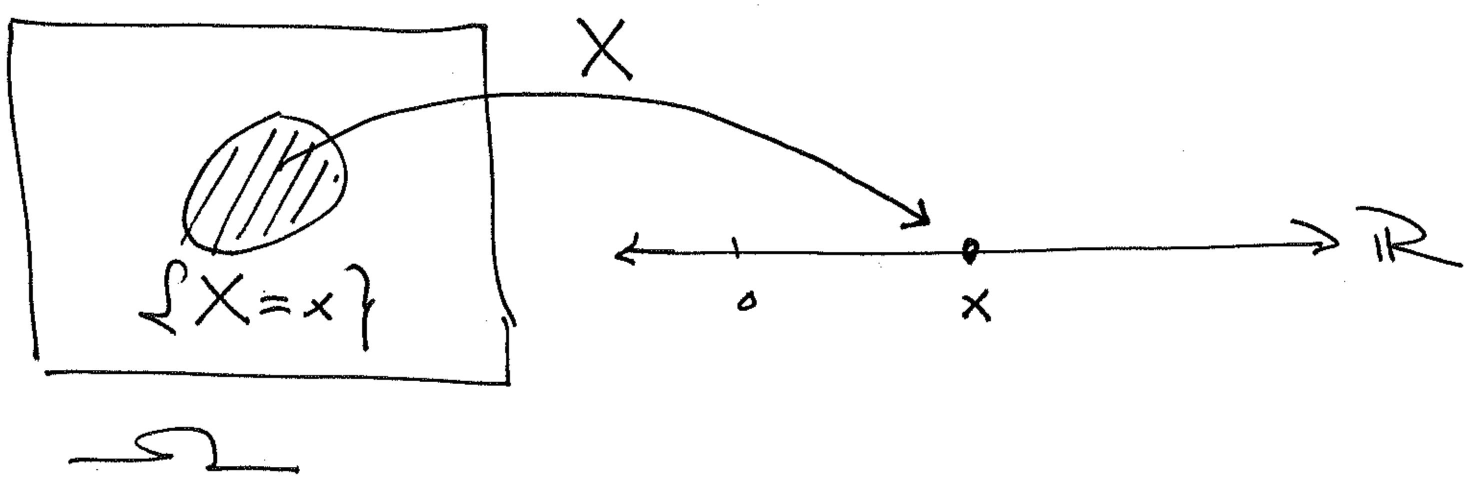
- use upper case for r.v.'s X
- use lower case for values of a r.v. x

$$X(\omega) = x$$

- denote by $\{X = x\}$ the event

$$\{X=x\} = \{\omega \in \Omega \mid X(\omega)=x\} \subseteq \Omega$$

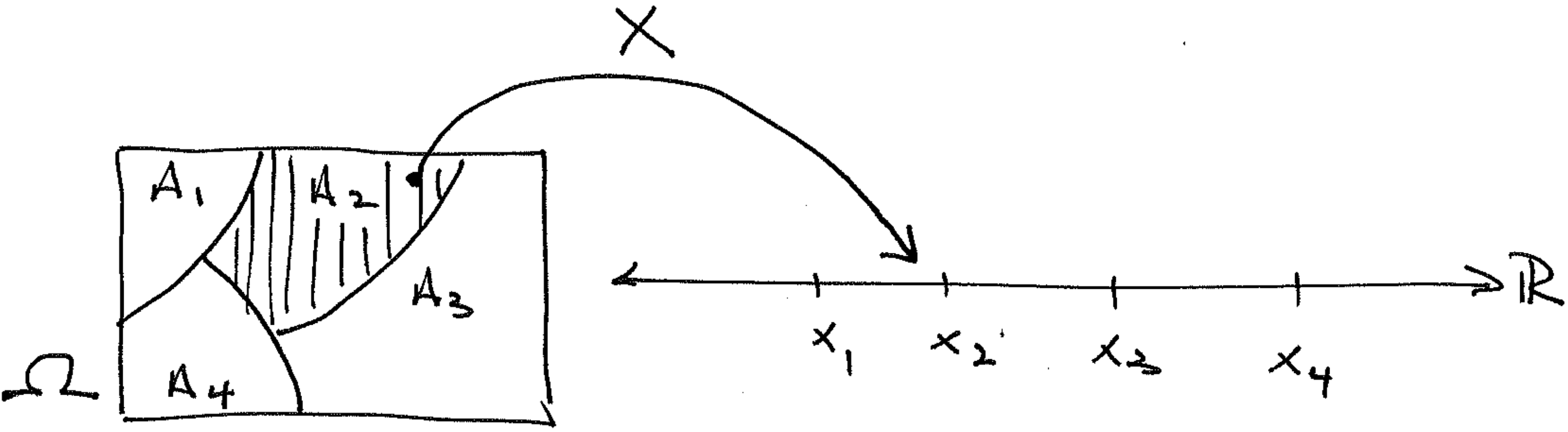
<u>Called</u>: preimage of x under X.



- think of X as mapping 'mass' from Ω to $\mathbb{R}$.

- X is called a <u>discrete r.v.</u> iff its range is a discrete subset of $\mathbb{R}$

$$\text{range}(X) = \{ X(\omega) \in \mathbb{R} \mid x \in \Omega \} \subseteq \mathbb{R}$$



here $A_i = \{ X = x_i \}$

- X is called a continuous r.v. .... later chap. 3.

- There are r.v.s that are neither discrete nor continuous.

<u>Ex.</u> (discrete r.v.)

Roll 2 fair independent dice.

Let $X((i,j)) = i+j$ (sum).

$$\Omega = \left\{\begin{array}{cccccc} 11 & 12 & 13 & \cdots & & 16 \\ 21 & 22 & & \cdots & & 26 \\ \vdots & & & & & \\ 61 & 62 & & \cdots & & 66 \end{array}\right\} \to \overbrace{\{2,3,4,5,6,7,8,9,10,11,12\}}^{\text{range}}$$

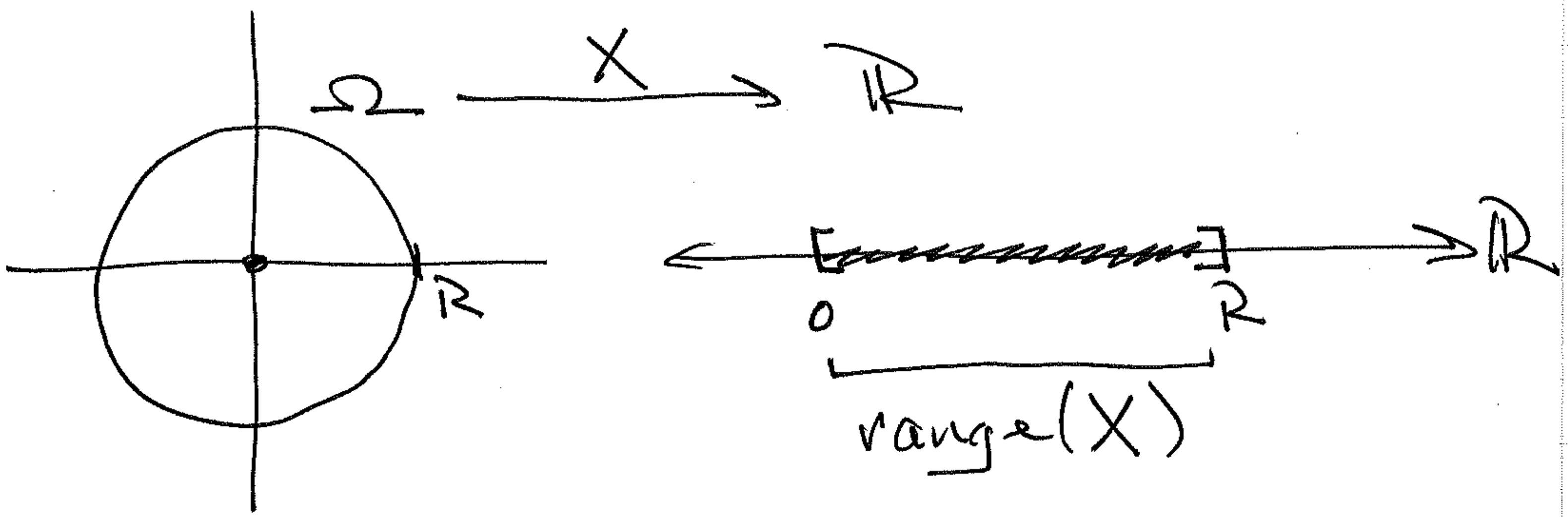
$$\Omega \xrightarrow[X]{} \mathbb{R}$$

<u>Ex.</u> (continuous r.v.)

Throw a dart at a circular target centered at $(0,0)$.

Let $X((a,b)) = \sqrt{a^2+b^2}$,

i.e. distance from dart (a,b) to $(0,0)$.



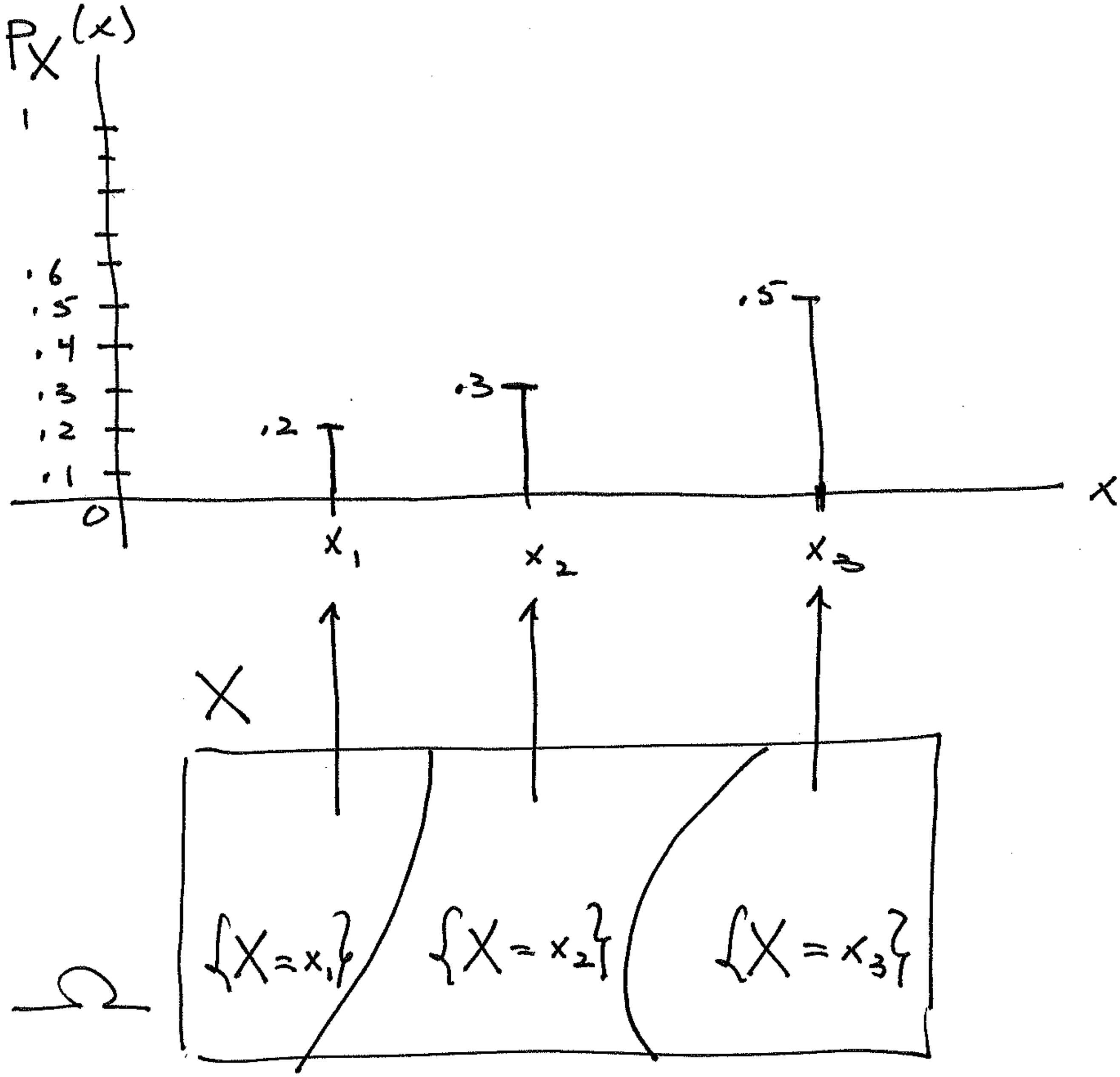
2.2 Probability mass functions

Let X be a discrete r.v. The <u>probability mass function (PMF)</u> of X is the function

$$p_X(x) = P(\{X=x\}) = P(X=x)$$

observe $p_X : \mathbb{R} \longrightarrow [0,1]$. Also note that necessarily

$$\sum_{x \in \mathrm{range}(X)} p_X(x) = 1$$



$$p_X(x) = \begin{cases} .2 & \text{if } x = x_1 \\ .3 & \text{if } x = x_2 \\ .5 & \text{if } x = x_3 \\ 0 & \text{otherwise} \end{cases}$$

<u>note</u>: $\sum_x p_X(x) = .2 + .3 + .5 = 1.0$

The Bernoulli random variable

Perform one Bernoulli trial. let

$$X(\omega) = \begin{cases} 1 & \text{if } \omega = \text{Success} \\ 0 & \text{if } \omega = \text{failure} \end{cases}$$

where probability of success is p, and failure is $1-p$. It's PMF is

$$P_X(k) = \begin{cases} p & \text{if } k = 1 \\ 1-p & \text{if } k = 0 \end{cases}$$

p is called the <u>Parameter</u> of the Bernoulli r.v.

The Binomial random Variable

Perform a sequence of n independent Bernoulli trials.

Let $X(\omega) = \#$ of successes.

The PMF is

$$p_X(k) = P(X=k)$$

$$= \binom{n}{k} p^k \cdot (1-p)^{n-k}$$

for $k = 0, 1, 2, \ldots, n$ (and $p_X(k) = 0$ otherwise)

check

$$\sum_{k=0}^{n} p_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= (\cancel{p} + (1-\cancel{p}))^n = 1^n = 1 \quad \checkmark$$

<u>note</u>. the Binomial r.v. has
2 Parameters:

$p = P(\text{success in any trial})$

$n = \#\text{ of trials}$.

<u>The Geometric random variable</u>

Perform identical independent Bernoulli trials (with parameter p). let X be the # trials until 1^{st} success.

what's the PMF?

$$p_X(k) = P(X = k) \qquad (k = 1, 2, 3, \ldots)$$

$=$ Probability of $(k-1)$ failures followed by 1 success so

$$P_X(k) = (1-p)^{k-1} \cdot p \qquad (k = 1, 2, 3, \ldots)$$

note:

$$\sum_{k=1}^{\infty} P_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} \cdot p$$

$$= p \cdot \sum_{k=0}^{\infty} (1-p)^k$$

$$= p \cdot \frac{1}{1-(1-p)} = \cancel{p} \cdot \frac{1}{\cancel{p}} = 1 \checkmark$$

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recall: $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ ✓ (geometric series with common ratio r.)

where $-1 < r < 1$.

Recall

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r} \qquad (r \neq 1)$$

Proof

$$S = \sum_{k=0}^{n-1} r^k$$

$$rS = \sum_{k=0}^{n-1} r^{k+1}$$

$$\therefore S - rS = \{r^0 + \cancel{r^1} + \cancel{\cdots} + \cancel{r^{n-1}}\}$$
$$- \{\cancel{r^1} + \cancel{r^2} + \cdots + \cancel{r^{n-1}} + r^n\}$$

$$S(1-r) = 1 - r^n$$

$$S = \frac{1-r^n}{1-r} \xrightarrow[\text{as } n \to \infty]{\lim} \frac{1}{1-r}$$

if $-1 < r < 1$