CSE 107 10-27-22

- hw 4 Due: 10 pm tonight.

Recall: Exponential r.v.

PDF $$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

<u>check:</u>

$$\int_{-\infty}^{\infty} f_X(x)\,dx = -\int_0^{\infty} -\lambda e^{-\lambda x}\,dx$$

$$= -e^{-\lambda x}\Big|_0^{\infty}$$

$$= -(0-1) = 1$$

also

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= -\int_0^{\infty} x\left(-\lambda e^{-\lambda x}\right) dx$$

$$\begin{cases} u = x & dv = -\lambda e^{-\lambda x} dx \\ du = dx & v = e^{-\lambda x} \end{cases}$$

$$= -\left(\cancelto{0}{x e^{-\lambda x} \Big|_0^{\infty}} + \frac{1}{\lambda}\int_0^{\infty} -\lambda e^{-\lambda x}\, dx \right)$$

$$= -\frac{1}{\lambda} \cdot e^{-\lambda x}\Big|_0^{\infty} = -\frac{1}{\lambda}(0-1) = \frac{1}{\lambda}$$

note: if X is exponential r.v.

then

$$P(X \geq a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \\ 1 & \text{if } a \leq 0 \end{cases}$$

If $a > 0$, then

$$P(X \geq a) = -\int_a^\infty -\lambda e^{-\lambda x}\, dx$$

$$= -e^{-\lambda x}\Big|_a^\infty$$

$$= -\left(0 - e^{-\lambda a}\right) = e^{-\lambda a}$$

Ex.

The time X until the next earthquake (4.0 or above) is modeled as an exp. r.v. with mean of 53 minutes.

What is the probability that such an earthquake occurs in next 10 minutes?

$$E[X] = \frac{1}{\lambda} = 53 \Rightarrow \lambda = \frac{1}{53}.$$

$$P(0 \leq X \leq 10) = P(X \geq 0) - P(X > 10)$$

$$= 1 - e^{-\frac{1}{53} \cdot 10}$$

$$= \boxed{.1719}$$

note:

$E[X]$ has units $\frac{\text{time}}{\text{event}}$

$\therefore$ $\frac{1}{\lambda}$ " " $\frac{\text{time}}{\text{event}}$

$\therefore$ λ " " $\frac{\text{event}}{\text{time}}$

i.e. λ is the "arrival rate".

3.2 Cumulative Distribution functions (CDF)

Defn

let X be a r.v. The CDF of X is

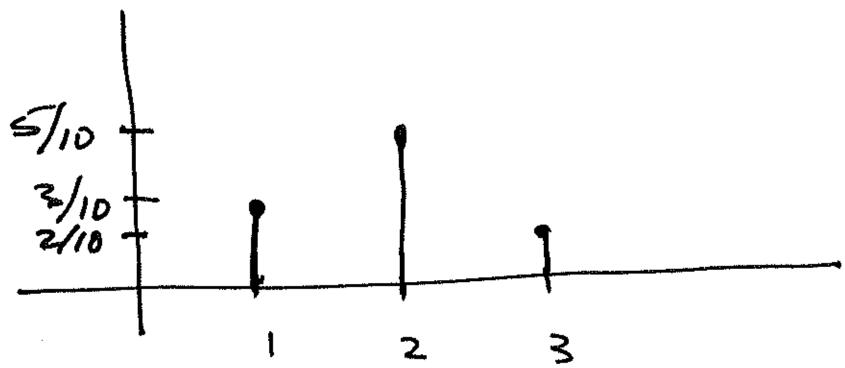
$$F_X(x) = P(X \le x)$$

The CDF 'accumulats' probability (mass) up to value $x \in \mathbb{R}$.

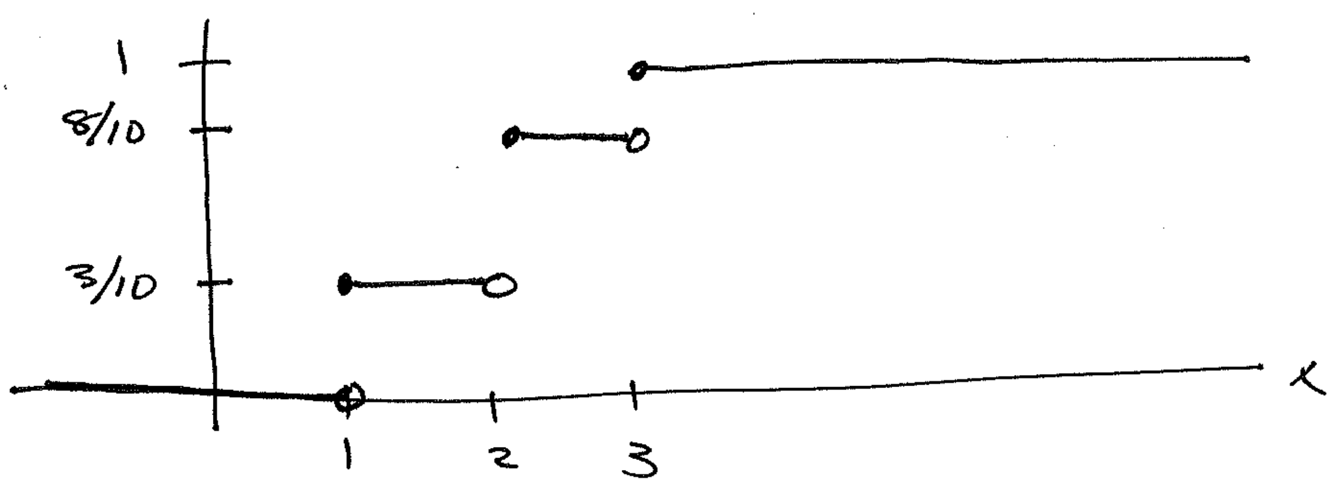
$$F_X(x) = \begin{cases} \sum_{k \le x} P_X(k) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{x} f_X(t)\, dt & \text{if } X \text{ is continuous} \end{cases}$$

Ex. (discrete)

PMF:
$$p_X(k) = \begin{cases} 3/10 & k=1 \\ 5/10 & k=2 \\ 2/10 & k=3 \\ 0 & \text{otherwise} \end{cases}$$



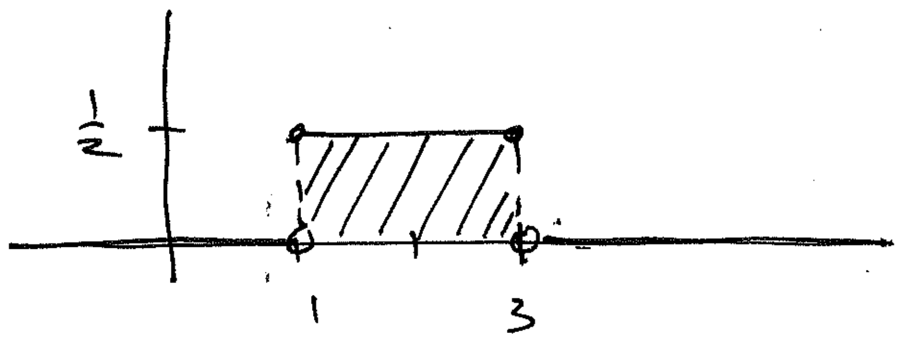
CDF:
$$F_X(x) = \begin{cases} 0 & \text{if } x<1 \\ 3/10 & \text{if } 1 \le x < 2 \\ 8/10 & \text{if } 2 \le x < 3 \\ 1 & \text{if } x \ge 3 \end{cases}$$



Ex. (continuous)

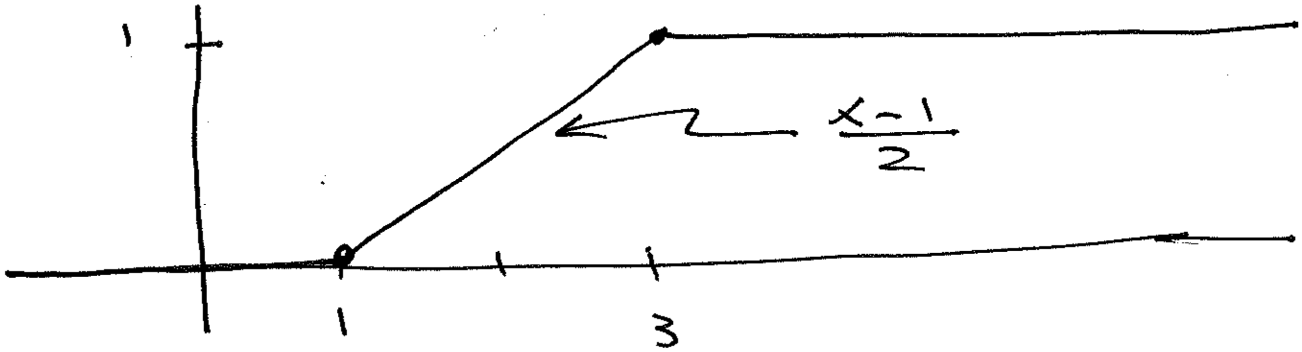
uniform on $[1, 3]$

PDF: $f_X(x) = \begin{cases} \frac{1}{2} & 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$



CDF: $F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{x-1}{2} & 1 \le x \le 3 \\ 1 & x > 3 \end{cases}$

for $x \in [1,3]$: $F_X(x) = \int_1^x \frac{1}{2}\, dt = \frac{1}{2} t \Big|_1^x = \frac{1}{2}(x-1)$



Properties

- A CDF: $F_X(x) = P(X \leq x)$

  is always monotonically nondecreasing

$$x \leq y \Rightarrow F_X(x) \leq F_X(y)$$

- If X is discrete $F_X(x)$ is piecewise constant

$$F_X(k) = \sum_{i=-\infty}^{k} p_X(i)$$

and

$$\begin{aligned} p_X(k) &= P(X = k) \\ &= P(X \leq k) - P(X \leq k-1) \\ &= F_X(k) - F_X(k-1) \end{aligned}$$

(assuming X has only integer values k).

- If X is continuous, then

$F_X(x)$ is a continuous function.

Also

$$F_X = \int_{-\infty}^{x} f_X(t)\,dt$$

and

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Ex.

You take a test 3 times, and your final score X is

$$X = \max(X_1, X_2, X_3)$$

where X_i is score on test $i = 1, 2, 3$.

suppose scores are independent, and uniform over $\{1, 2, 3, \ldots, 10\}$.

so

$$P_{X_i}(k) = \begin{cases} 1/10 & \text{if } k = 1, 2, \ldots, 10 \\ 0 & \text{otherwise} \end{cases}$$

find PMF of X.

<u>Soln</u>

first find CDF $F_X(x)$, then

PMF: $p_X(k) = F_X(k) - F_X(k-1)$.

so

$$F_X(k) = P(X \le k)$$
$$= P(X_1 \le k, X_2 \le k, X_3 \le k)$$

$$= P(X_1 \le k) \cdot P(X_2 \le k) \cdot P(X_3 \le k)$$

$$= F_{X_1}(k) \cdot F_{X_2}(k) \cdot F_{X_3}(k)$$

$$= \left(\frac{k}{10}\right) \cdot \left(\frac{k}{10}\right) \cdot \left(\frac{k}{10}\right)$$

$$= \left(\frac{k}{10}\right)^3$$

$$\therefore \quad P_X(k) = \left(\frac{k}{10}\right)^3 - \left(\frac{k-1}{10}\right)^3 \qquad \blacksquare$$

compare exponential & geometric r.v.s by comparing CDF.

Recall: if X is geometric, then

<u>PMF</u>: $P_X(k) = \begin{cases} p(1-p)^{k-1} & \text{if } k = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$

CDF: $F_{geo}(n) = \sum_{k=1}^{n} p(1-p)^{k-1}$

$$= \cancel{p} \cdot \frac{1-(1-p)^n}{\cancel{1-(1-p)}}$$

$$= 1-(1-p)^n$$

If Y is exponential (Param. λ)

PDF: $f_Y(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

so

CDF: $F_{exp}(x) = \int_0^x \lambda e^{-\lambda t}\, dt$

$$= -e^{-\lambda t}\Big|_0^x$$

$$= -\left(e^{-\lambda x} - 1\right)$$

$$= 1 - e^{-\lambda x} \quad (\text{if } x \geq 0)$$

To compare F_{exp} to F_{geo} pick $\delta > 0$

so that

$$\boxed{e^{-\lambda\delta} = 1 - p}$$

i.e. $p = 1 - e^{-\lambda\delta}$

$$-\lambda\delta = \ln(1-p)$$

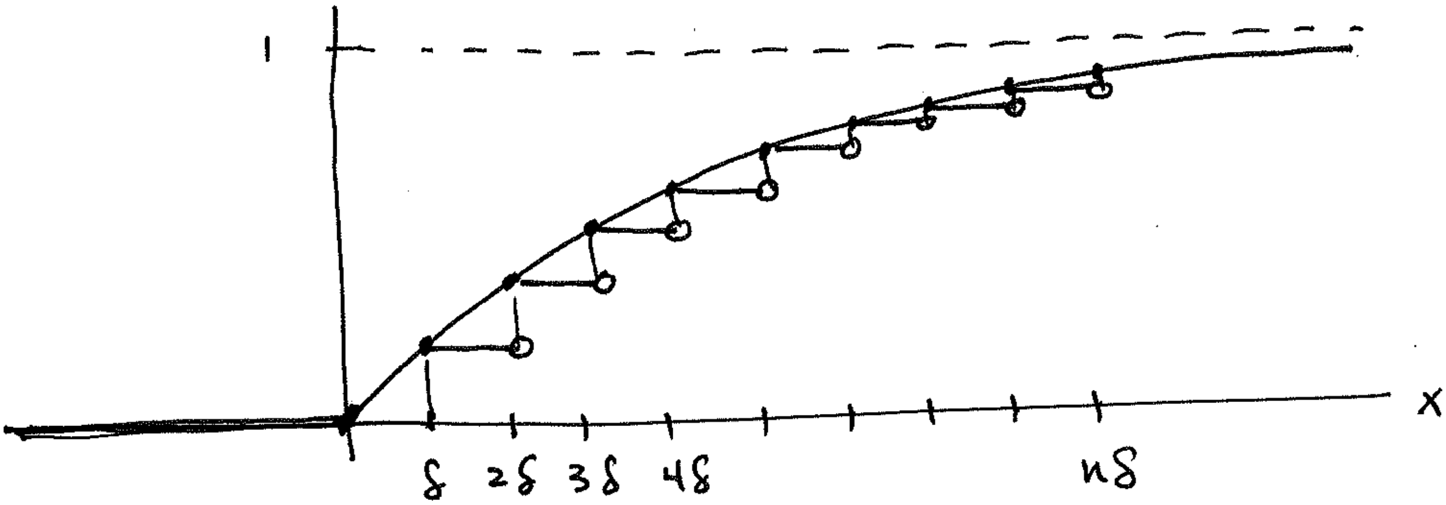
$$\therefore \quad \delta = \frac{-\ln(1-p)}{\lambda}$$

For this δ, we have

$$\left(e^{-\lambda\delta}\right)^n = (1-p)^n$$

$$1 - e^{-\lambda\delta n} = 1 - (1-p)^n$$

$$\therefore \quad F_{exp}(\delta n) = F_{geo}(n)$$



If we toss coin 1 every δ seconds with Prob of head

$$p = 1 - e^{-\lambda \delta},$$

then the 1st time of head approximates the exponential CDF with Parameter λ.