CSE 107 10-25-22

Ex. Alice passes through 4 traffic light on way to work, each equally likely to be red or green.

(a) find PMF, mean, variance of $X = \#$ red lights.

Solu:

X is binomial with $n=4$ and $p=\frac{1}{2}$.

$$P_X(k) = \binom{4}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{4-k}$$

$$= \frac{4!}{k!(4-k)!}\cdot\left(\frac{1}{2}\right)^4$$

$$= \frac{24}{k!(4-k)!}\cdot\frac{1}{16} = \frac{3}{2\,k!(4-k)!}$$

$$E[X] = np = 4 \cdot \frac{1}{2} = \boxed{2}$$

$$Var(X) = np(1-p) = 4 \cdot \frac{1}{2} \cdot \frac{1}{2} = \boxed{1}$$

(b) Suppose Alice is delayed 2 min. on each red light. Find variance of her commute time.

<u>Soln</u>

Let T = commute time. Then

$$T = T_0 + 2X$$

$\uparrow$ time with no red lights.

so $Var(T) = Var(2X)$

$$= 2^2 Var(X)$$

$$= 2^2 \cdot 1 = 4$$

also $\sigma_T = 2$

Recall

$$Var(X) = E[(X - E[X])^2]$$

3.1 Continuous r.v., PDF.

Defn

A random variable $X : \Omega \to \mathbb{R}$ is called continuous iff there exists a function

$$f_X : \mathbb{R} \to \mathbb{R}$$

called the probability density function (PDF) of X, such that for any subset $S \subseteq \mathbb{R}$:

$$P(X \in S) = \int_S f_X(x)\,dx$$

In particular

$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

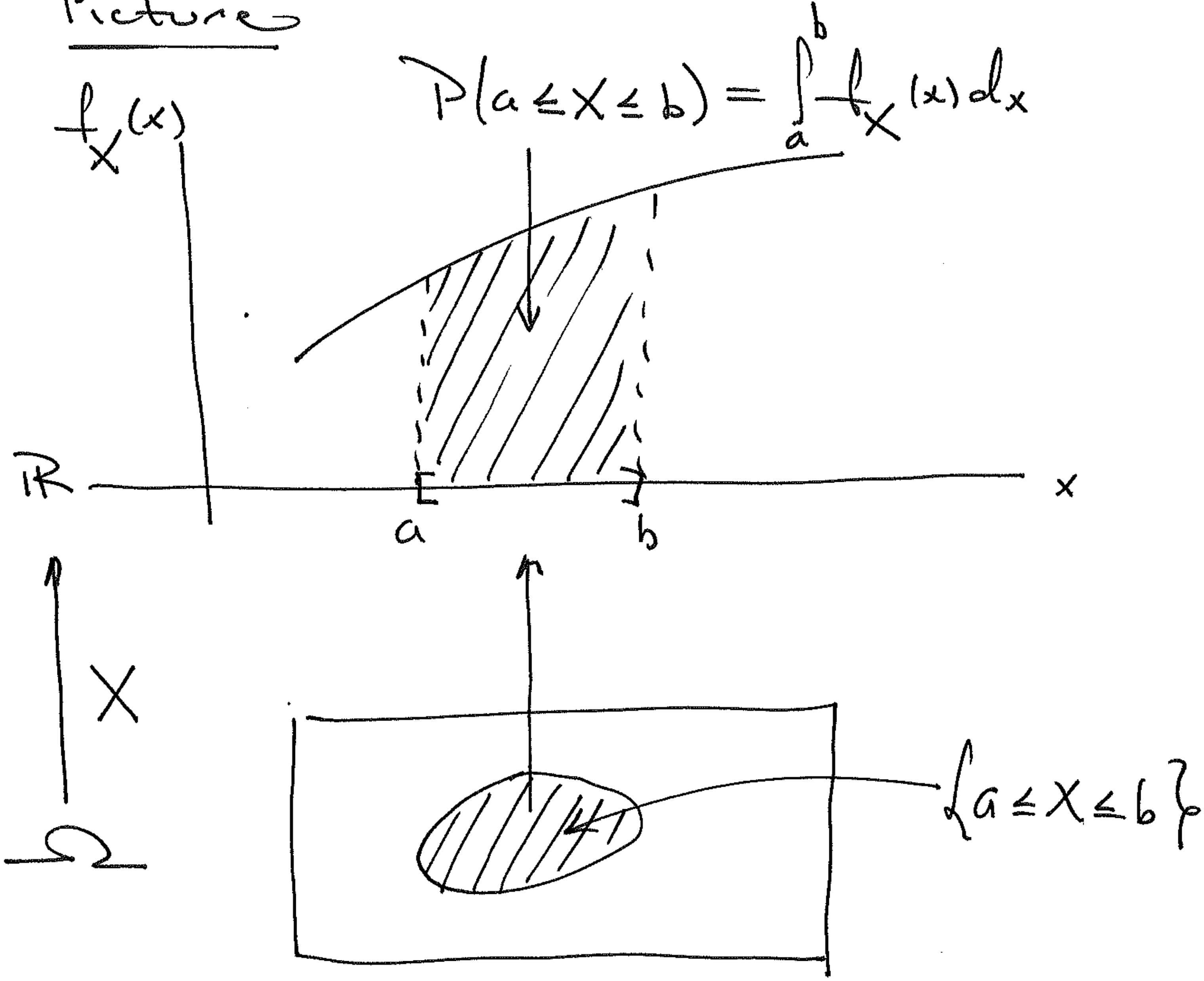
Note: integrating $[a,b]$, (a,b), $[a,b)$ or $(a,b]$ are all the same since

$$P(X=a) = \int_a^a f_X(x)\,dx = 0$$

Note: to be a valid PDF, must have $f_X(x) \ge 0$ for all $x \in \mathbb{R}$, and

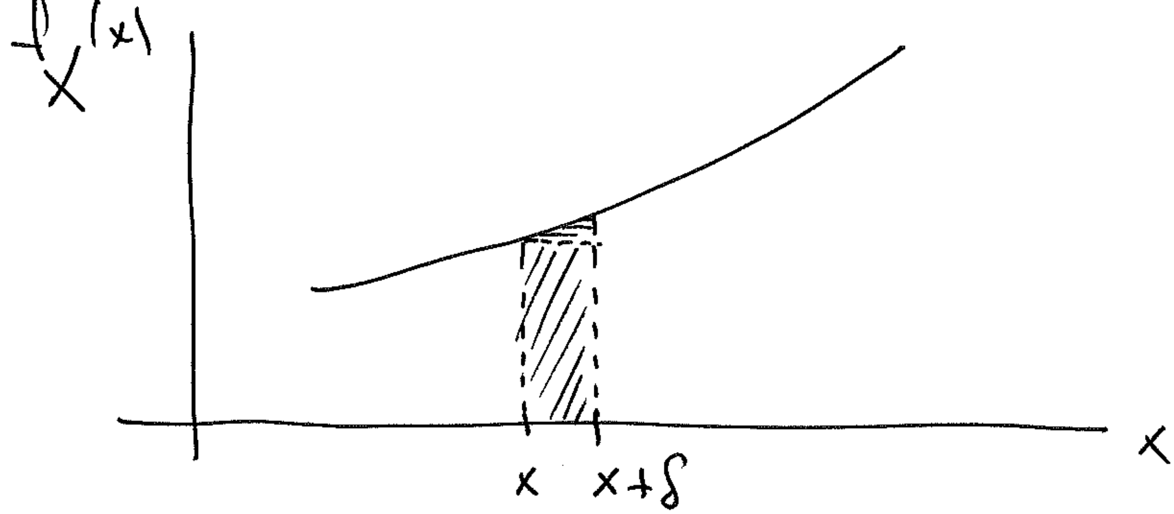
$$\int_{-\infty}^{\infty} f_X(x)\,dx = P(X \in \mathbb{R}) = P(\Omega) = 1.$$

<u>Picture</u>



<u>Note</u>: unlike the PMF, the PDF does not give 'mass', instead it gives 'mass density'.

let $\delta > 0$ be a small positive number.



$$\underbrace{P(x \le X \le x+\delta)}_{\text{units: Probability or 'mass'}} = \int_x^{x+\delta} f_X(t)\,dt \approx \underbrace{f_X(x)}\cdot \underbrace{\delta}_{\text{units: length}}$$

$\therefore$ units on $f_X(x)$ is 'mass per unit length'.

$\therefore$ $f_X(x)$ gives Probability density.

Ex. continuous uniform r.v. on $[a, b]$

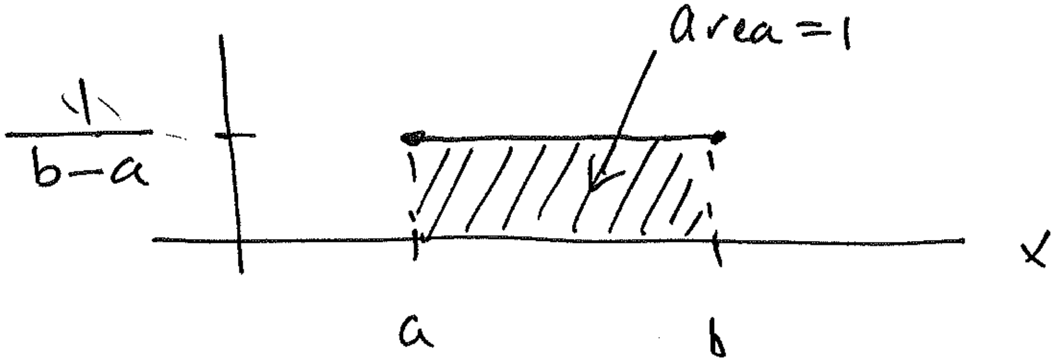
$$f_X(x) = \begin{cases} c & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

c can be determined from

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1,$$

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_a^b c\,dx = cx\Big|_a^b = c(b-a)$$

$$\therefore \quad c = \frac{1}{b-a}$$



$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Ex.

Alice's driving time to work is 40-44 minutes in good weather and 44-50 minutes in bad. Uniformly distributed in both cases. Suppose

$$P(\text{good}) = \frac{2}{3}.$$

Determine the PDF of X, Alice's commute time.

Soln

$$f_X(x) = \begin{cases} c_1 & \text{if } 40 \le x \le 44 \\ c_2 & \text{if } 44 < x \le 50 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{2}{3} = P(\text{good}) = \int_{40}^{44} c_1\, dx = c_1(44-40) = 4c_1$$

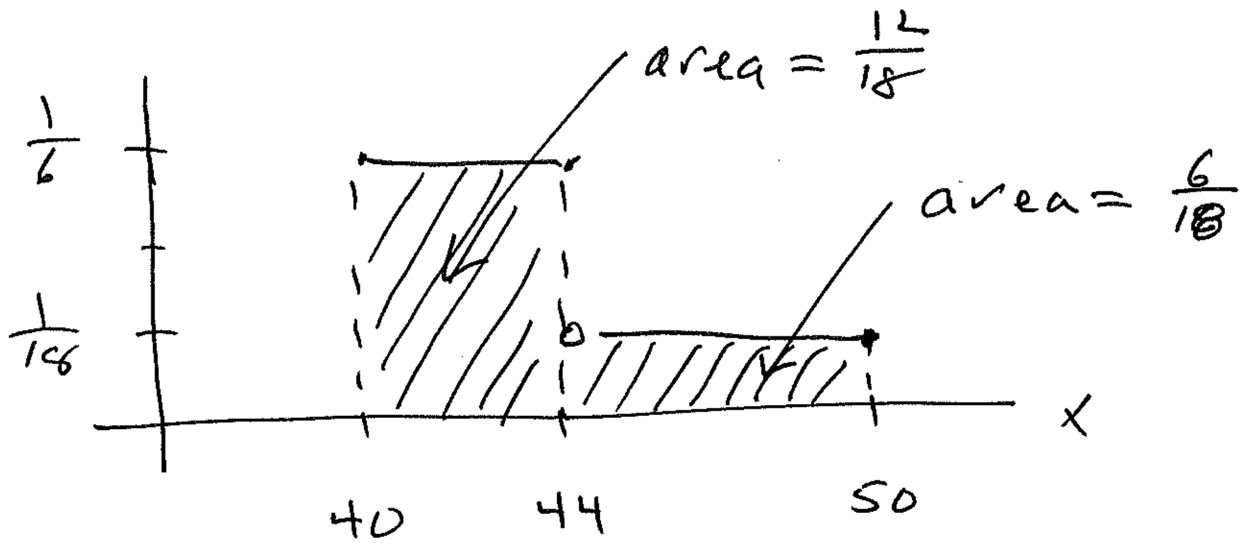
$$\therefore \quad c_1 = \frac{2}{12} = \frac{1}{6}$$

$$\frac{1}{3} = P(\text{bad}) = \int_{44}^{50} c_2\, dx = c_2(50-44) = 6c_2$$

$$\therefore \quad c_2 = \frac{1}{18}$$

$\therefore$

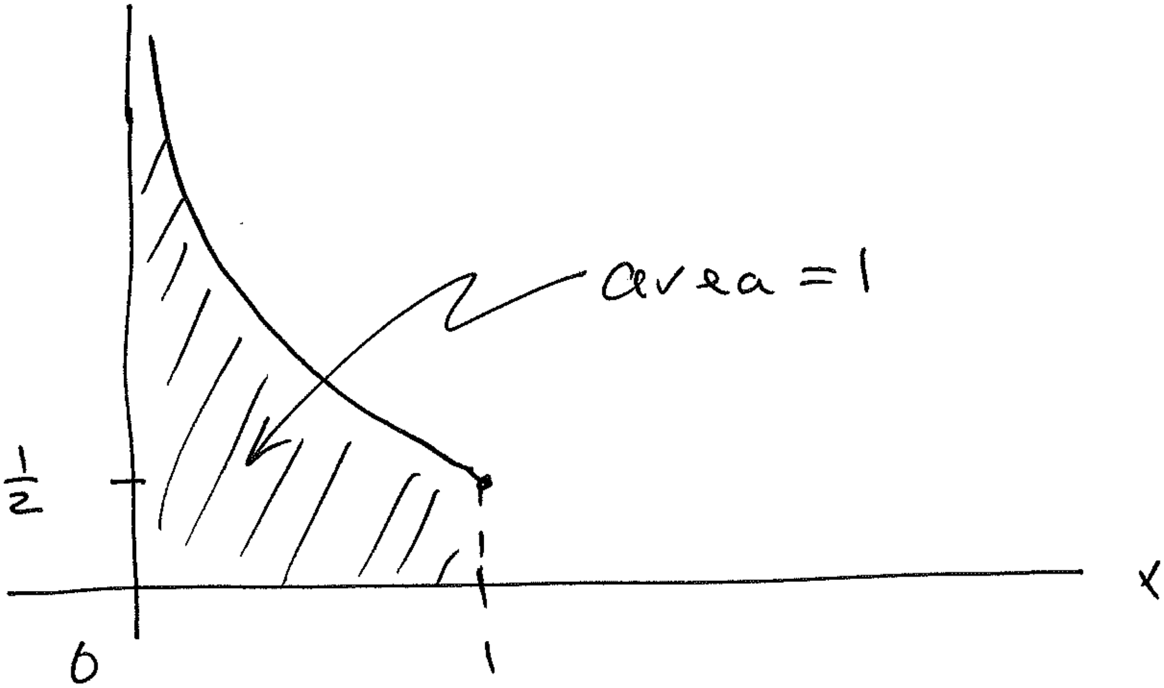
$$f_X(x) = \begin{cases} 1/6 & \text{if } 40 \le x \le 44 \\ 1/18 & 44 < x \le 50 \\ 0 & \end{cases}$$



<u>Ex.</u>

Suppose X has PDF

$$f_X(x) = \begin{cases} \dfrac{1}{2\sqrt{x}} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\int_0^1 \frac{1}{2\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1 \tfrac{1}{2}x^{-\frac{1}{2}}\,dx$$

$$= \cancel{\tfrac{1}{2}}\lim_{a\to 0^+} \left.\frac{x^{1/2}}{\cancel{1/2}}\right|_a^1$$

$$= \lim_{a\to 0^+}\left(\sqrt{1}-\sqrt{a}\right) = \boxed{1}$$

<u>Defn</u>

The <u>expected value</u> of X is

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

As before, we have an <u>expected value rule</u> for functions $g(X)$:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \, f_X(x) \, dx$$

The n^{th} moment of X is

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

The variance is

$$\begin{aligned} Var(X) &= E\left[(X - E[X])^2\right] \\ &= E[X^2] - E[X]^2 \end{aligned}$$

exercise ! Prove this

If $Y = aX + b$, then

$$E[Y] = aE[X] + b$$

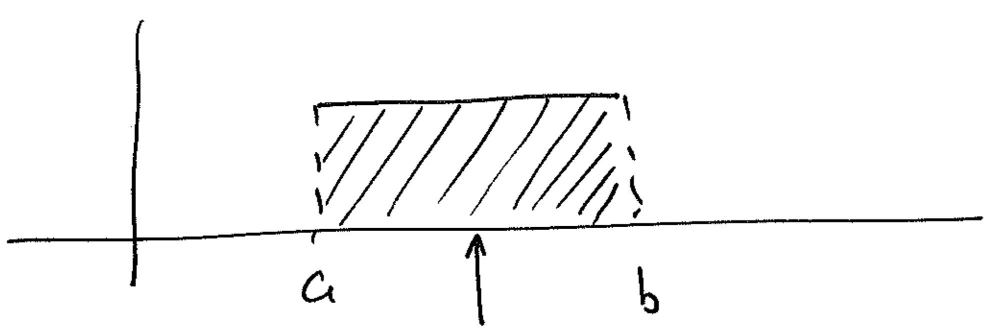
and

$$Var[Y] = a^2 Var(X)$$

Exercise: prove these.

Ex. cont. uniform r.v. on $[a,b]$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



$E[X] = \frac{a+b}{2}$

so

$$E[X] = \int_a^b x\left(\frac{1}{b-a}\right)dx = \left(\frac{1}{b-a}\right)\int_a^b x\,dx$$

$$= \frac{1}{b-a}\cdot\frac{x^2}{2}\Big|_a^b = \frac{1}{b-a}\cdot\frac{1}{2}(b^2-a^2)$$

$$= \frac{1}{2}\cdot\frac{\cancel{(b-a)}(b+a)}{\cancel{(b-a)}} = \frac{a+b}{2}.$$

check:

$$E[X^2] = \frac{a^2+ab+b^2}{3}$$

and

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

Exercise

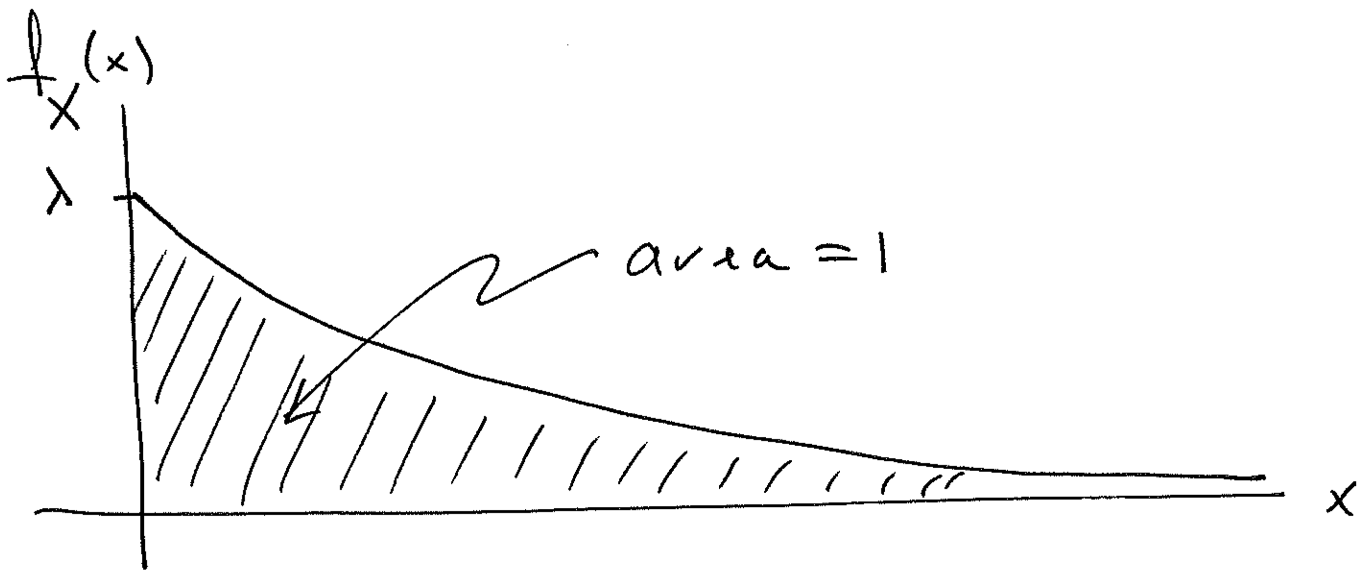
Alice driving time X again:

$$f_X(x) = \begin{cases} 1/6 & 40 \le x \le 44 \\ 1/18 & 44 \le x \le 50 \\ 0 & \text{otherwise} \end{cases}$$

$$\left.\begin{aligned} E[X] &= \frac{131}{3} \\ E[X^2] &= \frac{17{,}228}{9} \\ Var(X) &= \frac{67}{9} \end{aligned}\right\} \text{ verify !!}$$

EX. Exponential random variable X with parameter $\lambda > 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



check: $\displaystyle\int_0^\infty \lambda e^{-\lambda x}\, dx = 1$.

check: $E[X] = \frac{1}{\lambda}$, $Var(X) = \frac{1}{\lambda^2}$