ese 107 10-18-22

- hw3 due Thur.
- lab2 Posted.

Mean and Variance of Geometric r.v.

Recall if X is geometric, Parameter p, then

$$P_X(k) = (1-p)^{k-1} p \qquad (k = 1, 2, 3 \ldots)$$

≥ 0

- $E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p$

- $E[X^2] = \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p$

- $Var(X) = E[X^2] - E[X]^2$

find $E[X]$ using total expectation

let

$A_1 = \{X = 1\} = \{1^{st} \text{ flip is head}\}$

and

$A_2 = \{X > 1\} = \{1^{st} \text{ flip is tail}\}$

$\therefore P(A_1) = p, \; P(A_2) = 1-p$

Also $E[X \mid A_1] = 1$, and

$$E[X \mid A_2] = E[1 + X]$$
$$= 1 + E[X]$$

Thus

$$E[X] = P(A_1) \cdot E[X \mid A_1] + P(A_2) \cdot E[X \mid A_2]$$

$$= p \cdot 1 + (1-p)(1 + E[X])$$

$$\cancel{E[X]} = \cancel{p} + 1 - \cancel{p} - pE[X] + \cancel{E[X]}$$

$$\therefore \quad pE[X] = 1$$

$$\therefore \quad \boxed{E[X] = \frac{1}{p}.}$$

Similarly

$$E[X^2 \mid A_1] = 1$$

and

$$E[X^2 \mid A_2] = E[(1+X)^2]$$

$$= E[1 + 2X + X^2]$$

$$= 1 + 2E[X] + E[X^2]$$

Thus

$$E[X^2] = p \cdot 1 + (1-p)\left(1 + 2 \cdot \frac{1}{p} + E[X^2]\right]$$

$$E[\cancel{X^2}] = \cancel{p} + 1 + \frac{2}{p} + E[\cancel{X^2}] - \cancel{p} - 2 - pE[X^2]$$

$$pE[X^2] = -1 + \frac{2}{p}$$

$$\therefore E[X^2] = \frac{2}{p^2} - \frac{1}{p}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= \left(\frac{2}{p^2} - \frac{1}{p}\right) - \frac{1}{p^2}$$

$$= \frac{1}{p^2} - \frac{1}{p}$$

$$\boxed{Var(X) = \frac{1-p}{p^2}}$$