CSE 107 10-13-22

- midterm 1 : Tue 10/18
  lecture to follow

2.6 Conditioning

Given $X : \Omega \to \mathbb{R}$ a r.v.
the PMF was

$$p_X(x) = P(X = x)$$

we can do this with any Prob. law,
not just $P(\cdot)$. Do this for
$P(\cdot \mid A)$.

Defn

let $X : \Omega \to \mathbb{R}$ be an r.v.,

$A \subseteq \Omega$ with $P(A) > 0$.

The conditional PMF of X

given A is

$$\boxed{p_{X|A}(x) = P(X = x \mid A)}$$

note

$$p_{X|A}(x) = \frac{P(X = x, A)}{P(A)}$$

note the events

$$\{X = x\} \cap A \quad \text{for } x \in \text{range}(X)$$

are disjoint, with union A.

$$\therefore \; P(A) = \sum_x P(\{X=x\} \cap A)$$

so

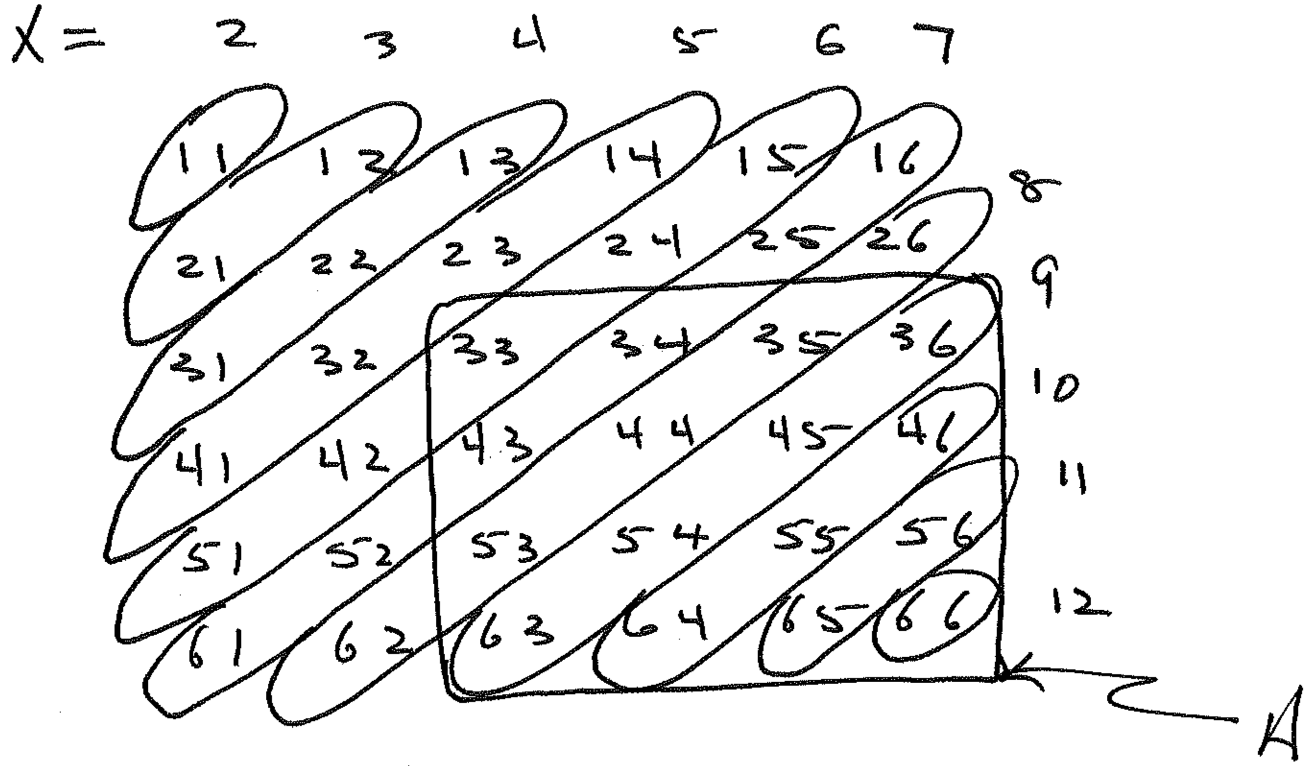
$$\sum_x P_{X|A}(x) = \frac{\sum_x P(\{X=x\} \cap A)}{P(A)}$$

$$= \frac{P(A)}{P(A)} = 1,$$

as expected (and required) of a PMF.

Ex.

Roll 2 independent, fair 6-sided dice, and let X be sum of top faces.



$$
p_X(k) = \begin{cases} 1/36 & \text{if } k = 2, 12 \\ 2/36 & k = 3, 11 \\ 3/36 & k = 4, 10 \\ 4/36 & k = 5, 9 \\ 5/36 & k = 6, 8 \\ 6/36 & k = 7 \\ 0 & \text{otherwise} \end{cases}
$$

Let $A = \{\min \geq 3\}$. $P(A) = \frac{16}{36}$

$$P_{X|A}(k) = \begin{cases} 1/16 & k = 6, 12 \\ 2/16 & k = 7, 11 \\ 3/16 & k = 8, 10 \\ 4/16 & k = 9 \\ 0 & \text{otherwise} \end{cases}$$

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<u>Ex.</u>

flip a weighted coin, $P(H) = p$, until 1^{st} head. Let $X = \#$ flips. Then X is geometric with parameter p.

$$P_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k = 1, 2, 3, \ldots\ldots \\ 0 & \text{otherwise} \end{cases}$$

Let A be the event that the 1^{st} head occurs with n flips.

$$P(A) = \sum_{k=1}^{n} P_X(k)$$

$$= \sum_{k=1}^{n} (1-p)^{k-1} \cdot p$$

$$= p \cdot \sum_{k=0}^{n-1} (1-p)^k$$

$$= \cancel{p} \cdot \frac{1-(1-p)^n}{\cancel{1-(1-p)}} = 1-(1-p)^n$$

Another way:

$$P(A) = 1 - P(A^c)$$

$$= 1 - (1-p)^n$$

Thus

$$P_{X|A}(k) = \begin{cases} \dfrac{(1-p)^{k-1} \cdot p}{1-(1-p)^n} & k = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

Now let Y be another r.v. such that

$$P_Y(y) = P(Y=y) > 0$$

for all $y \in \text{range}(Y)$. we can now condition on events of form

$$A = \{Y = y\}$$

Defn

The conditional PMF of X given Y is

$$\boxed{\begin{aligned} p_{X|Y}(x|y) &= P(X=x \mid Y=y) \\ &= \frac{P(X=x, Y=y)}{P(Y=y)} \end{aligned}}$$

Think of y as a fixed constant and x as variable.

note:

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

Thus

$$\boxed{P_{X,Y}(x,y) = P_Y(y) \cdot P_{X|Y}(x|y)}$$

Also

$$\boxed{P_{X,Y}(x,y) = P_X(x) \cdot P_{Y|X}(y|x)}$$

Ex.

A professor answers questions wrongly with prob. $1/4$, independent of other questions. The # of questions asked is 0, 1, 2 with eq. prob. $\frac{1}{3}$.

let

X = # questions asked

and

Y = # questions answered wrongly.

find $P_{X,Y}(x,y)$. We have

$$P_X(x) = \begin{cases} 1/3 & x = 0,1,2 \\ 0 & \text{otherwise} \end{cases}$$

$$p_{Y|X}(y|x) = \begin{cases} \left[1 \right. & \left. x=0,\ y=0 \right] \\ \left[3/4 \right. & x=1,\ y=0 \\ \left. 1/4 \right. & \left. x=1,\ y=1 \right] \\ \left[9/16 \right. & x=2,\ y=0 \\ 3/8 & x=2,\ y=1 \\ \left. 1/16 \right. & \left. x=2,\ y=2 \right] \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$p_{X,Y}(x,y) = \begin{cases} 1/3 & x=0,\ y=0 \\ 1/4 & \quad 1 \quad\quad 0 \\ 1/12 & \quad 1 \quad\quad 1 \\ 3/16 & \quad 2 \quad\quad 0 \\ 1/8 & \quad 2 \quad\quad 1 \\ 1/48 & \quad 2 \quad\quad 2 \\ 0 & \text{otherwise} \end{cases}$$

find marginal $P_Y(y)$.

$$P_Y(y) = \sum_x P_{X,Y}(x,y)$$

$$= \sum_x P_X(x) \cdot P_{Y|X}(y|x)$$

Thus

$$P_Y(y) = \begin{cases} 37/48 & y=0 \\ 10/48 & y=1 \\ 1/48 & y=2 \\ 0 & \text{otherwise} \end{cases}$$

<u>Exercise</u> find $P_{X|Y}(x|y)$

conditional Expectation

let X be a r.v., $A \subseteq \Omega$ with $P(A) > 0$. Defn:

$$\boxed{E[X|A] = \sum_x x \cdot P_{X|A}(x)}$$

also

$$\boxed{E[g(X)|A] = \sum_x g(x) \cdot P_{X|A}(x)}$$

we can condition on $A = \{Y = y\}$ provided $P(Y=y) > 0$.

$$\boxed{E[X \mid Y=y] = \sum_{x} x \cdot p_{X|Y}(x|y)}$$

Total expectation Theorem

Given a partition $A_1, A_2, \dots, A_n$ of Ω, then

$$E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X \mid A_i]$$

Proof

by total prob. thm:

$$p_X(x) = \sum_{i=1}^{n} P(A_i) \cdot p_{X|A_i}(x)$$

mult. by x, sum over $x \in \underline{\text{range}}(X)$

$$E[X] = \sum_x x \cdot p_X(x)$$

$$= \sum_x x \cdot \sum_{i=1}^{n} P(A_i) \cdot p_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i) \cdot \sum_x x \cdot p_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i) \cdot E[X|A_i]$$

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Mean and Variance of Geometric

recall geom. with param. p

$$p_X(k) = (1-p)^{k-1} \cdot p \quad (k = 1, 2, \ldots$$