ese 107 10-11-22

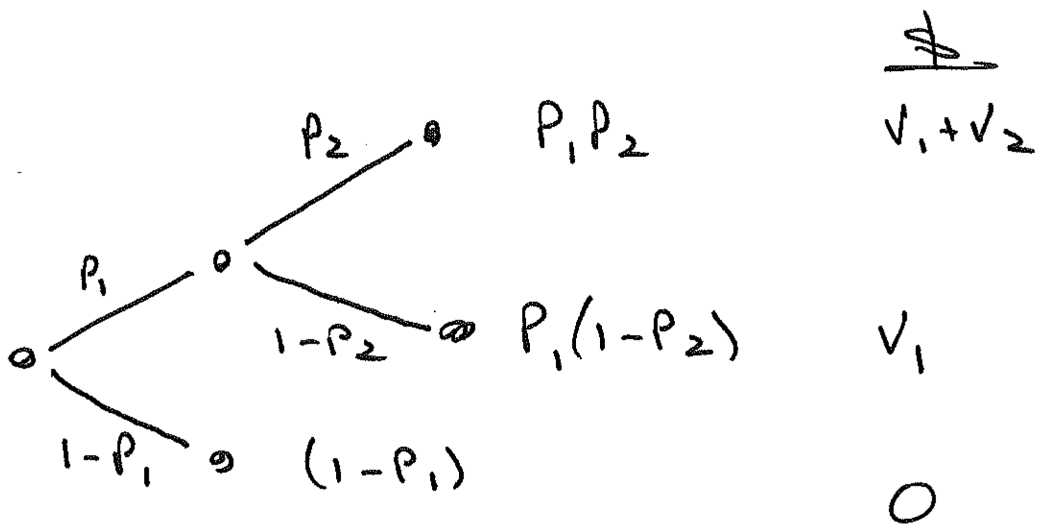
Ex. Quiz Problem

A game consists of 2 questions

	Q_1	Q_2	
P(correct)	p_1	p_2	$(\neq 0, 1)$
Payoff	v_1	v_2	($)

The 1st wrong answer terminates game.
Can answer in order (Q_1, Q_2) or (Q_2, Q_1). Which is preffered?

• let X = # \$ won with (Q_1, Q_2)



$$P_X(x) = \begin{cases} p_1 p_2 & \text{if } x = v_1 + v_2 \\ p_1(1-p_2) & x = v_1 \\ 1-p_1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = (v_1+v_2)p_1p_2 + v_1 p_1 (1-p_2)$$

$$= \cancel{v_1 p_1 p_2} + v_2 p_1 p_2 + v_1 p_1 - \cancel{v_1 p_1 p_2}$$

$$= \boxed{v_1 p_1 + v_2 p_1 p_2}$$

- let $Y = \#\$$ won with (Q_2, Q_1)

$$P_Y(y) = \begin{cases} P_1 P_2 & \text{if } y = v_1 + v_2 \\ P_2(1-P_1) & y = v_2 \\ 1 - P_2 & y = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y] = \boxed{V_2 P_2 + V_1 P_1 P_2}$$

- when should we choose (Q_1, Q_2) over (Q_2, Q_1).

$$V_1 P_1 + V_2 P_1 P_2 > V_2 P_2 + V_1 P_1 P_2$$

$$V_1 P_1 - V_1 P_1 P_2 > V_2 P_2 - V_2 P_1 P_2$$

$$V_1 P_1 (1 - P_2) > V_2 P_2 (1 - P_1)$$

$$\frac{V_1 P_1}{1 - P_1} > \frac{V_2 P_2}{1 - P_2}$$

Exercise

do this for 3 questions. Does it generalize?

2.5 Joint PMFs, Multiple r.v.s

Defn

Given 2 r.v.s X, Y their joint PMF is

$$p_{X,Y}(x,y) = \underbrace{P(X=x, Y=y)}_{\displaystyle \shortparallel}$$

$$P(\{X=x\} \cap \{Y=y\})$$

If $A \subseteq \text{range}(X) \times \text{range}(Y)$,

then

$$P\big((x,y)\in A\big) = \sum_{(x,y)\in A} p_{X,Y}(x,y)$$

<u>Ex.</u>

Suppose $\text{range}(X) = \{1,2,3,4\}$ and $\text{range}(Y) = \{1,2,3\}$. let joint PMF be

$p_X(x)$

4/20	5/20	6/20	5/20

$p_Y(y)$

Y \ X	1	2	3	4	$p_Y(y)$
3	1/20	1/20	2/20	1/20	5/20
2	2/20	3/20	3/20	1/20	9/20
1	1/20	1/20	1/20	3/20	6/20

$p_{X,Y}(x,y)$

Suppose $A = \{(1,3), (2,1), (4,2)\}$,

then

$$P((X,Y) \in A) = \frac{1}{20} + \frac{1}{20} + \frac{1}{20} = \frac{3}{20}$$

find $P(Y=3)$. notice that

disjoint union

$$\{X=1\} \cup \{X=2\} \cup \{X=3\} \cup \{X=4\} = \Omega$$

$$\therefore P(X=1, Y=3) + P(X=2, Y=3) + P(X=3, Y=3) + P(X=4, Y=3)$$
$$= P(Y=3)$$

$$\therefore P_Y(3) = P_{X,Y}(1,3) + P_{X,Y}(2,3) + P_{X,Y}(3,3) + P_{X,Y}(4,3)$$

$$\therefore P_Y(3) = \sum_x P_{X,Y}(x,3)$$

Thus we can recover $p_X(x)$ and $p_Y(y)$ from $p_{X,Y}(x,y)$

$$\boxed{p_X(x) = \sum_y p_{X,Y}(x,y)} \leftarrow$$ marginal PMFs

$$\boxed{p_Y(y) = \sum_x p_{X,Y}(x,y)} \leftarrow$$

<u>Functions of multiple random vars.</u>

Let $Z = g(X,Y)$, where

$$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

Then

$$p_Z(z) = \sum_{\{(x,y) \mid g(x,y)=z\}} p_{X,Y}(x,y)$$

The <u>expected value rule</u> can be generalized

$$\boxed{\begin{aligned} E[Z] &= E[g(X,Y)] \\ &= \sum_x \sum_y g(x,y)\, p_{X,Y}(x,y) \end{aligned}}$$

<u>Exercise</u>: prove this.

Ex. Show

$$E[aX+bY+c] = aE[X] + bE[Y] + c$$

Proof

$$E[aX+bY+c] = \sum_x \sum_y (ax+by+c)\, p_{X,Y}(x,y)$$

$$= a\sum_x \sum_y x\, p_{X,Y}(x,y) + b\sum_x \sum_y y\, p_{X,Y}(x,y) + c\sum_x \sum_y p_{X,Y}(x,y)$$

$$= a\sum_x x\cdot\left(\sum_y p_{X,Y}(x,y)\right) + b\sum_y y\cdot\left(\sum_x p_{X,Y}(x,y)\right) + c$$

$$= a\sum_x x\, p_X(x) + b\sum_y y\, p_Y(y) + c$$

$$= aE[X] + bE[Y] + c.$$ ∎

Ex. (previous)

$$E[X+3Y] = E[X] + 3E[Y]$$

$$= \left(1\cdot\frac{4}{20} + 2\cdot\frac{5}{20} + 3\cdot\frac{6}{20} + 4\cdot\frac{5}{20}\right)$$

$$+ 3\left(1\cdot\frac{6}{20} + 2\cdot\frac{9}{20} + 3\cdot\frac{5}{20}\right)$$

$$= \boxed{\frac{169}{20}}$$

more than 2 r.v.s

we can do all this for 3, 4, 5, ...
random variables

Suppose X, Y, Z are r.v.s. their joint PMF is

$$P_{X,Y,Z}(x,y,z) = P(X=x, Y=y, Z=z)$$

3 marginal PMFs:

$$P_{X,Y}(x,y) = \sum_z P_{X,Y,Z}(x,y,z)$$

$$P_{X,Z}(x,z) = \sum_y P_{X,Y,Z}(x,y,z)$$

$$P_{Y,Z}(y,z) = \sum_x P_{X,Y,Z}(x,y,z)$$

expected value rule

$$\boxed{E[g(X,Y,Z)] = \sum_{x,y,z} g(x,y,z)\, p_{X,Y,Z}(x,y,z)}$$

Now suppose we have n r.v.s

$$X_1, X_2, X_3, \ldots, X_n$$

Then

$$E[a_1X_1 + a_2X_2 + \cdots + a_nX_n]$$

$$= a_1E[X_1] + \cdots + a_nE[X_n].$$

by e.v.r.

the mean of the Binomial r.v

let X be a Binomial r.v. with Parameters n, p. Recall

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad (k = 0, 1, \ldots, n)$$

want

$$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = \; ? \quad \boxed{np} \; !!!$$

Note $X = X_1 + X_2 + \cdots + X_n$ where each X_i is Bernoulli w. param p,

i.e $X_i = \begin{cases} 1 & i^{th} \text{ flip is heads} \\ 0 & \text{" " " tails} \end{cases}$

$$\therefore E[X] = E[X_1 + \cdots + X_n]$$

$$= E[X_1] + \cdots + E[X_n]$$

$$= p + \cdots + p$$

$$= \boxed{np}$$

Also

$$\mathrm{Var}(X) = \boxed{np(1-p)} \ \ldots \text{ later}$$