

CSE 107**Probability and Statistics for Engineers****Approximation of the Binomial PMF by the Poisson PMF**

Suppose that $n \geq 1$ is large, $p \in (0, 1)$ is small and

$$\lambda = np.$$

We prove that under these conditions, the Poisson PMF with parameter λ is a good approximation to the Binomial PMF with parameters n and p . In other words, for any k in the range $0 \leq k \leq n$, we have

$$e^{-\lambda} \cdot \frac{\lambda^k}{k!} \approx \binom{n}{k} p^k (1-p)^{n-k}.$$

More precisely, we set $p = \lambda/n$ and prove the limit

$$(1) \quad \lim_{n \rightarrow \infty} \left[\binom{n}{k} \left(\frac{\lambda}{n} \right)^k \left(1 - \left(\frac{\lambda}{n} \right) \right)^{n-k} \right] = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

We begin by establishing the following limits, which hold for any $\lambda > 0$ and $k \geq 0$:

$$(2) \quad \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n} \right)^n = e^{-\lambda}$$

and

$$(3) \quad \lim_{n \rightarrow \infty} \left(\frac{n!}{(n-k)! (n-\lambda)^k} \right) = 1.$$

Proof of (2):

Take the natural log of the sequence, and observe

$$\ln \left(1 - \frac{\lambda}{n} \right)^n = n \cdot \ln \left(1 - \frac{\lambda}{n} \right) = \frac{\ln \left(1 - \frac{\lambda}{n} \right)}{\frac{1}{n}}.$$

By l'Hopital's rule we obtain

$$\lim_{n \rightarrow \infty} \left[\ln \left(1 - \frac{\lambda}{n} \right)^n \right] = \lim_{n \rightarrow \infty} \left[\frac{\frac{1}{(1-\lambda/n)} \cdot \frac{\lambda}{n^2}}{-\frac{1}{n^2}} \right] = \lim_{n \rightarrow \infty} \left(\frac{-\lambda}{1-\lambda/n} \right) = -\lambda.$$

Since $e^x = \exp(x)$ is a continuous function, it follows that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n &= \lim_{n \rightarrow \infty} \exp \left[\ln \left(1 - \frac{\lambda}{n}\right)^n \right] \\
&= \exp \left(\lim_{n \rightarrow \infty} \left[\ln \left(1 - \frac{\lambda}{n}\right)^n \right] \right) \\
&= \exp(-\lambda) = e^{-\lambda},
\end{aligned}$$

establishing the claim. ■

Proof of (3):

By canceling factors top and bottom we write

$$\begin{aligned}
\frac{n!}{(n-k)!(n-\lambda)^k} &= \frac{n(n-1)(n-2) \cdots (n-k+1)}{(n-\lambda)^k} \\
&= \left(\frac{n}{n-\lambda}\right) \left(\frac{n-1}{n-\lambda}\right) \left(\frac{n-2}{n-\lambda}\right) \cdots \left(\frac{n-k+1}{n-\lambda}\right).
\end{aligned}$$

For each i in the range $0 \leq i \leq k-1$,

$$\lim_{n \rightarrow \infty} \left(\frac{n-i}{n-\lambda}\right) = \lim_{n \rightarrow \infty} \left(\frac{1 - \frac{i}{n}}{1 - \frac{\lambda}{n}}\right) = 1,$$

and therefore

$$\lim_{n \rightarrow \infty} \left(\frac{n!}{(n-k)!(n-\lambda)^k}\right) = 1 \cdot 1 \cdot 1 \cdots 1 = 1,$$

as required. ■

Proof of (1):

We have

$$\begin{aligned}
\binom{n}{k} \cdot \left(\frac{\lambda}{n}\right)^k \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k} &= \frac{n!}{k!(n-k)!} \cdot \left(\frac{\lambda}{n}\right)^k \cdot \frac{(1 - (\lambda/n))^n}{(1 - (\lambda/n))^k} \\
&= \frac{n!}{k!(n-k)!} \cdot \left(\frac{\lambda/n}{1 - (\lambda/n)}\right)^k \cdot \left(1 - \frac{\lambda}{n}\right)^n \\
&= \frac{n!}{k!(n-k)!} \cdot \left(\frac{\lambda}{n-\lambda}\right)^k \cdot \left(1 - \frac{\lambda}{n}\right)^n
\end{aligned}$$

$$= \frac{n!}{(n-k)! (n-\lambda)^k} \cdot \frac{\lambda^k}{k!} \cdot \left(1 - \frac{\lambda}{n}\right)^n.$$

Equations (2) and (3) now yield

$$\lim_{n \rightarrow \infty} \left[\binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \left(\frac{\lambda}{n}\right)\right)^{n-k} \right] = 1 \cdot \frac{\lambda^k}{k!} \cdot e^{-\lambda} = e^{-\lambda} \cdot \frac{\lambda^k}{k!},$$

which proves equation (1). ■