CSE 102
Homework Assignment 1
Solutions

1. (Problem 3.1-1) Let $f(n)$ and $g(n)$ asymptotically positive functions. Prove that $f(n)+g(n)=\Theta(\max(f(n),\ g(n)))$.

   Proof:
   Since $f(n)$ and $g(n)$ are asymptotically positive, we know that there exists a positive constant n_0 such that $f(n)>0$ and $g(n)>0$ for all $n\ge n_0$. For any such n we have

   $$\begin{aligned} 0 &\le \max\big(f(n),g(n)\big) \\ &\le \min\big(f(n),g(n)\big)+\max\big(f(n),g(n)\big) \\ &\le 2\cdot\max\big(f(n),g(n)\big). \end{aligned}$$

   But $f(n)+g(n)=\min\big(f(n),g(n)\big)+\max\big(f(n),g(n)\big)$, and therefore

   $$0\le 1\cdot\max\big(f(n),g(n)\big)\le f(n)+g(n)\le 2\cdot\max\big(f(n),g(n)\big)$$

   for all $n\ge n_0$, showing that $f(n)+g(n)=\Theta(\max\,(f(n),g(n)))$. ∎

2. Prove or disprove: If $f(n)=\Theta(g(n))$, then $f(n)^2=\Theta(g(n)^2)$.

   Proof:
   Since $f(n)=\Theta(g(n))$, there are positive constants c_1, c_2 and n_0 such that

   $$0\le c_1g(n)\le f(n)\le c_2g(n)$$

   for all $n\ge n_0$. Squaring this inequality gives $0\le c_1^2g(n)^2\le f(n)^2\le c_2^2g(n)^2$, showing that $f(n)^2=\Theta(g(n)^2)$. ∎

3. Prove or disprove: If $f(n)=\Theta(g(n))$, then $2^{f(n)}=\Theta(2^{g(n)})$.

   Counter Example:
   Let $f(n)=2n$ and $g(n)=n$. Then $f(n)=\Theta(g(n))$, but $2^{2n}=4^n=\omega(2^n)$, and therefore $2^{f(n)}=\omega(2^{g(n)})$, whence $2^{f(n)}\ne\Theta(2^{g(n)})$. ∎

4. Let $f(n)$ and $g(n)$ be asymptotically positive functions, and assume that $\lim\limits_{n\to\infty}g(n)=\infty$. Prove that if $f(n)=\Theta(g(n))$, then $\ln\big(f(n)\big)=\Theta\big(\ln\big(g(n)\big)\big)$.

   Proof:
   Assume $f(n)=\Theta(g(n))$. Then there exist positive constants c_1, c_2 and n_0 such that

   $$0\le c_1g(n)\le f(n)\le c_2g(n)$$

for all $n \geq n_0$. Since $\lim_{n\to\infty} g(n) = \infty$, the constant n_0 can be chosen large enough so that

$$1 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

for all $n \geq n_0$. Take ln() of all terms in the preceding inequality to get

$$0 \leq \ln(c_1) + \ln\big(g(n)\big) \leq \ln\big(f(n)\big) \leq \ln(c_2) + \ln\big(g(n)\big)$$

for all $n \geq n_0$. Since $\lim_{n\to\infty} g(n) = \infty$, the term $\ln\big(g(n)\big)$ dominates the constants $\ln(c_1)$ and $\ln(c_2)$, whence $\ln(c_1) + \ln\big(g(n)\big) = \Omega\Big(\ln\big(g(n)\big)\Big)$ and $\ln(c_2) + \ln\big(g(n)\big) = O\Big(\ln\big(g(n)\big)\Big)$. The desired result $\ln\big(f(n)\big) = \Theta\Big(\ln\big(g(n)\big)\Big)$ now follows by Exercise 4 on page 4 of the handout on asymptotic growth rates. ■

5. (Problem 3.2-8) Show that if $f(n)\ln f(n) = \Theta(n)$, then $f(n) = \Theta(n/\ln n)$. Hint: use the result of the preceding problem.

   Proof:
   Assume $f(n)\ln f(n) = \Theta(n)$. Since $\lim_{n\to\infty} n = \infty$, we can, by the result of problem (4), take ln () of both sides of this equation to get

   $$\ln\big(f(n)\big) + \ln\Big(\ln\big(f(n)\big)\Big) = \Theta(\ln(n)).$$

   Since the term $\ln\Big(\ln\big(f(n)\big)\Big) = o(\ln\big(f(n)\big)$, we can, by Exercise 9g on page 8 of the handout on asymptotic growth rates, drop the lower order term and write this as $\ln\big(f(n)\big) = \Theta(\ln(n))$. Substituting this into our assumption gives $f(n)\Theta(\ln n) = \Theta(n)$, and therefore

   $$f(n) = \Theta(n) \cdot \frac{1}{\Theta(\ln n)} = \Theta(n) \cdot \Theta\left(\frac{1}{\ln n}\right) = \Theta\left(\frac{n}{\ln n}\right)$$

   where we have used Exercise 11bc on page 8 of the handout on asymptotic growth rates. ■

6. Consider the statement: $f(cn) = \Theta(f(n))$.

   a. Determine a function $f(n)$ and a constant $c > 0$ for which the statement is false.

      Example:
      Let $f(n) = 2^n$ and $c = 2$. Then the statement says $2^{2n} = \Theta(2^n)$, i.e. $4^n = \Theta(2^n)$. This is false since $4^n = \omega(2^n)$ by Exercise 4e on page 7 of the handout on Asymptotic Growth Rates, and since $\omega(2^n) \cap \Theta(2^n) = \emptyset$ by Exercise 6 on page 5 of the same handout. ■

   b. Determine a function $f(n)$ for which the statement is true for all $c > 0$.

      Example:
      Let $f(n) = n$. Then the statement says $cn = \Theta(n)$. This is true for any $c > 0$ by Exercise 2 on page 3 of the handout on Asymptotic Growth Rates. ■

7. Determine the asymptotic order of the expression $\sum_{i=1}^{n} a^i$ where $a > 0$ is a constant, i.e. find a simple function $g(n)$ such that the expression is in the class $\Theta\big(g(n)\big)$. (Hint: consider the cases $a = 1$, $a > 1$, and $0 < a < 1$ separately.)

   Solution: In the case $a = 1$ we have $\sum_{i=1}^{n} a^i = n = \Theta(n)$. Now assume $a \neq 1$ and use the formula for the sum of a geometric series:

$$\sum_{i=1}^{n} a^i = a\left(\sum_{i=0}^{n-1} a^i\right) = a\left(\frac{a^n - 1}{a - 1}\right) = \left(\frac{a}{a-1}\right) \cdot a^n - \left(\frac{a}{a-1}\right)$$

   If $a > 1$ then $a^n \to \infty$ as $n \to \infty$, so the first term dominates the second, and $\sum_{i=1}^{n} a^i = \Theta(a^n)$. If $0 < a < 1$ then $a^n \to 0$, so the constant term (which is positive in this case) dominates, and therefore $\sum_{i=1}^{n} a^i = \Theta(1)$. ■

8. Use induction to prove that $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$ for all $n \geq 1$.

   Proof:

   I. $\sum_{k=1}^{1} k^4 = 1 = \frac{2\cdot 15}{30} = \frac{1\cdot(1+1)\cdot(6\cdot 1^3+9\cdot 1^2+1-1)}{30}$, and the base case is established.

   II. Let $n \geq 1$. Assume for this *n* that

$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$$

   We must show that

$$\sum_{k=1}^{n+1} k^4 = \frac{(n+1)(n+2)(6(n+1)^3+9(n+1)^2+(n+1)-1)}{30}$$

   We have

$$\begin{aligned}
\sum_{k=1}^{n+1} k^4 &= \left(\sum_{k=1}^{n} k^4\right) + (n+1)^4 \\
&= \frac{n(n+1)(6n^3+9n^2+n-1)}{30} + (n+1)^4 \qquad \text{(by the induction hypothesis)} \\
&= \frac{(n+1)\big[n(6n^3+9n^2+n-1)+30(n+1)^3\big]}{30} \\
&= \frac{(n+1)\big[6n^4+39n^3+91n^2+89n+30)\big]}{30} \\
&= \frac{(n+1)(n+2)\big[6n^3+27n^2+37n+15)\big]}{30} \\
&= \frac{(n+1)((n+1)+1)\big[6(n+1)^3+9(n+1)^2+(n+1)-1)\big]}{30}
\end{aligned}$$

   Thus the formula holds for $n + 1$ as well. The formula is valid for all *n* by induction. ■