CSE 102 1-9-20

Ex. Prove $\sqrt{n+10} = \Theta(\sqrt{n})$

Proof

we must show there exist positive c_1, c_2, n_0 s.t. $\forall n \geq n_0$:

$$0 \leq c_1\sqrt{n} \leq \sqrt{n+10} \leq c_2\sqrt{n}.$$

Let $c_1 = 1$, $c_2 = \sqrt{2}$, $n_0 = 10$.

Then if $n \geq n_0 = 10$, we have

$$-10 \leq 0 \quad \text{and} \quad 10 \leq n$$

$\therefore$ $-10 \leq (1-1)n$ and $10 \leq (2-1)n$

$\therefore$ $-10 \leq (1-c_1^2)n$ and $10 \leq (c_2^2-1)n$

$\therefore$

$$c_1^2 n \le n+10 \quad \text{and} \quad n+10 \le c_2^2 n$$

$$\therefore \quad 0 \le c_1^2 n \le n+10 \le c_2^2 n$$

$$\therefore \quad 0 \le c_1\sqrt{n} \le \sqrt{n+10} \le c_2\sqrt{n} .$$ ∎

<u>Exercise</u>

find other constants c_1, c_2, n_0 that work.

<u>Exercise</u>

show that $c_1 = \sqrt{\frac{1}{2}}$, $c_2 = \sqrt{\frac{3}{2}}$, $n_0 = 20$

Also work.

Exercise

Let $a, b \in \mathbb{R}$ where $b > 0$.
Show that

$$(n+a)^b = \Theta(n^b).$$

Theorem

If $h(n) = O(g(n))$ and $f(n) \le h(n)$
for all suff. large n, then

$$f(n) = O(g(n))$$

Proof.

Since $h(n) = O(g(n))$, we have positive c_1, n_1 s.t.

(1) $\forall n \geq n_1 : \; 0 \leq h(n) \leq c_1 g(n)$

Since $f(n) \leq h(n)$ for suff. large n, we have pos. n_2 s.t.

(2) $\forall n \geq n_2 : \; 0 \leq f(n) \leq h(n)$

We must find positive c_3, n_3 s.t.

(3) $\forall n \geq n_3 : \; 0 \leq f(n) \leq c_3 g(n)$.

Let $c_3 = c_1$, and $n_3 = \max(n_1, n_2)$.

Thus c_3, n_3 are positive and

(1) and (2) $\Rightarrow$ (3). ∎

<u>Exercise</u>

If $h(n) = \Omega(g(n))$ and

$f(n) \geq h(n)$ for all suff. large n,

then

$$f(n) = \Omega(g(n))$$

Exercise

If $h_2(n) = O(g(n))$, $h_1(n) = \Omega(g(n))$

and if

$$h_1(n) \le f(n) \le h_2(n)$$

for all suff. large n, then

$$f(n) = \Theta(g(n)).$$

Ex.

Let $k > 0$. Prove that

$$\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$$

Remark:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\vdots$$

Proof

$$\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1} = O(n^{k+1})$$

Also

$$\sum_{i=1}^{n} i^K \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^K$$

$$\geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \left\lceil \frac{n}{2} \right\rceil^K$$

$$= \left(n - \left\lceil \tfrac{n}{2} \right\rceil + 1\right) \left\lceil \frac{n}{2} \right\rceil^K$$

$$= \left(\left\lfloor \tfrac{n}{2} \right\rfloor + 1\right) \left\lceil \frac{n}{2} \right\rceil^K$$

$$> \left(\frac{n}{2} - \cancel{1} + \cancel{1}\right) \left(\frac{n}{2}\right)^K \qquad \left\{ \begin{array}{l} \lfloor x \rfloor > x - 1 \\ \lceil x \rceil \geq x \end{array} \right.$$

$$= \left(\frac{n}{2}\right)^{K+1}$$

$$= \left(\frac{1}{2}\right)^{K+1} \cdot n^{K+1} = \Omega\left(n^{K+1}\right)$$

hence $\sum_{i=1}^{n} i^K = \Theta\left(n^{K+1}\right)$.

Defn

$o(g(n))$

$= \{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) < c\,g(n) \}$

write: $f(n) = o(g(n))$

say: "$g(n)$ is a strict asympt. u.b. for $f(n)$"

Recall:

$O(g(n)) =$

$\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq c\,g(n) \}$

Obviously:

$o(g(n)) \subseteq O(g(n))$

Lemma 1

$$f(n) = o(g(n)) \text{ iff } \lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = 0$$

Prove:

$f(n) = o(g(n))$ iff

$$\forall c>0, \exists n_0>0, \forall n \geq n_0: \; 0 \leq f(n) < c\,g(n)$$

$$\therefore \; \forall c>0, \exists n_0>0, \forall n \geq n_0: \; 0 \leq \frac{f(n)}{g(n)} < c \, ,$$

This is the definition of

$$\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = 0 \, .$$

Ex. $\ln(n) = o(n)$

$$\lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right) \underset{\text{l'Hop.}}{=} \lim_{n\to\infty}\frac{1/n}{1} = 0 \qquad \blacksquare$$

Exercise show $\log_b(n) = o(n)$ for any $b > 1$.

Ex. let $k > 0$. show $n^k = o(e^n)$

$$\lim_{n\to\infty}\left(\frac{n^k}{e^n}\right) = \lim_{n\to\infty}\left(\frac{k\cdot n^{k-1}}{e^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{k\cdot(k-1)n^{k-2}}{e^n}\right)$$

$$\vdots$$

$$= 0 \qquad \text{after } \lceil k \rceil \text{ app. of l'Hop}$$

Exercise

show $n^k = o(b^n)$ for any $b>1$.

Exercise show

$o(g(n)) \cap \Omega(g(n)) = \phi$

Defn

$\omega(g(n)) =$

$\{f(n) \mid \forall c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c\,g(n) < f(n)\}$

check: $\omega(g(n)) \subseteq \Omega(g(n))$

Exercise:

- $f(n) = \omega(g(n))$ iff $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = \infty$

- $\omega(g(n)) \cap O(g(n)) = \phi$

Picture: