CSE 102 1-7-20

Asymptotic growth of functions:

Defn

A function $f(n)$ is asymptotically non-negative (a.n.n.) iff

$$\exists\, n_0 > 0, \forall n \geq n_0 : f(n) \geq 0$$

$f(n)$ is asymptotically positive (a.p.) iff

$$\exists\, n_0 > 0, \forall n \geq n_0 : f(n) > 0$$

Defn

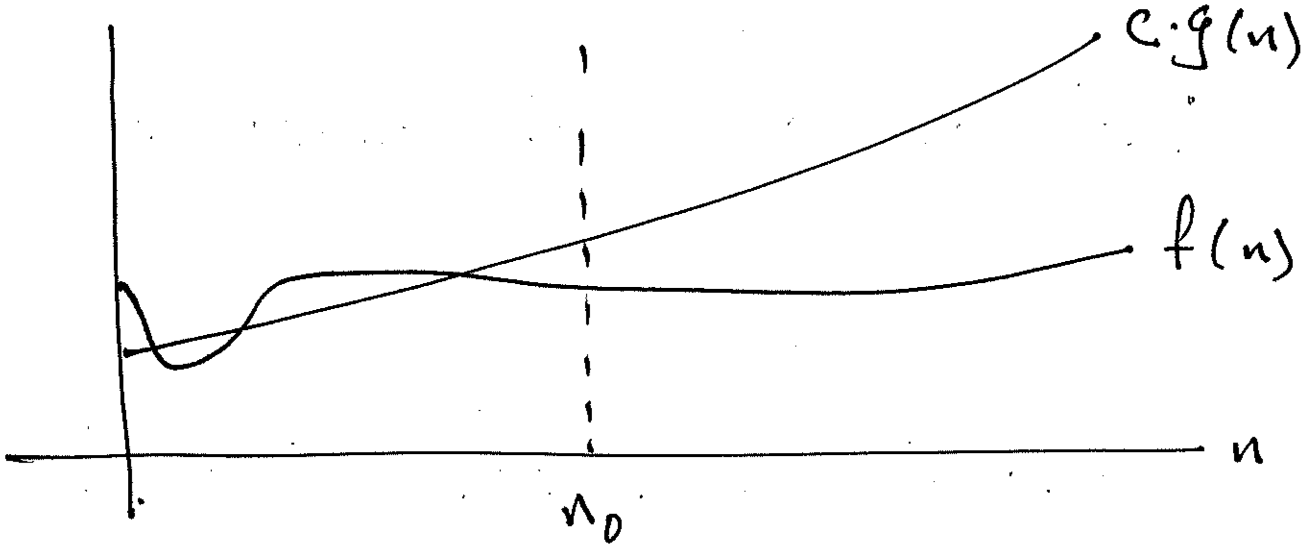
let $f(n), g(n)$ be functions

$$O(g(n)) = \{f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}$$

we say "$f(n)$ is order $g(n)$",

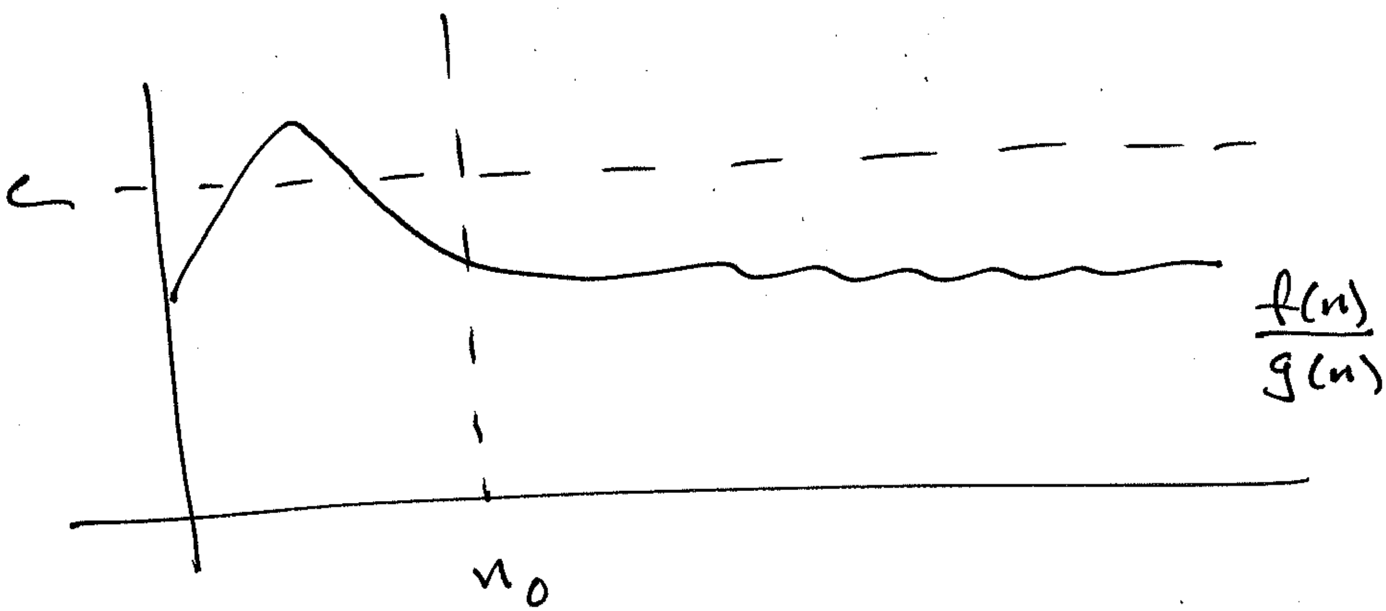
we write $\boxed{f(n) = O(g(n))}$ to

mean $f(n) \in O(g(n))$



<u>note</u>: if $g(n)$ is a.p. this is equivalent to

$$\exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq \frac{f(n)}{g(n)} \leq c$$



<u>Fact</u>: if $p(n), q(n)$ are polynomials, both a.n.n. (positive lead coefficient), and if

$$\deg p(n) \leq \deg q(n)$$

then

$$p(n) = O(q(n))$$

<u>note</u>: 5 classes of asym. growth

"analogy"

$$f(n) = O(g(n)) \quad \sim \quad x \leq y$$

$$f(n) = \Omega(g(n)) \quad \sim \quad x \geq y$$

$$f(n) = \Theta(g(n)) \quad \sim \quad x = y$$

$$f(n) = o(g(n)) \quad \sim \quad x < y$$

$$f(n) = \omega(g(n)) \quad \sim \quad x > y$$

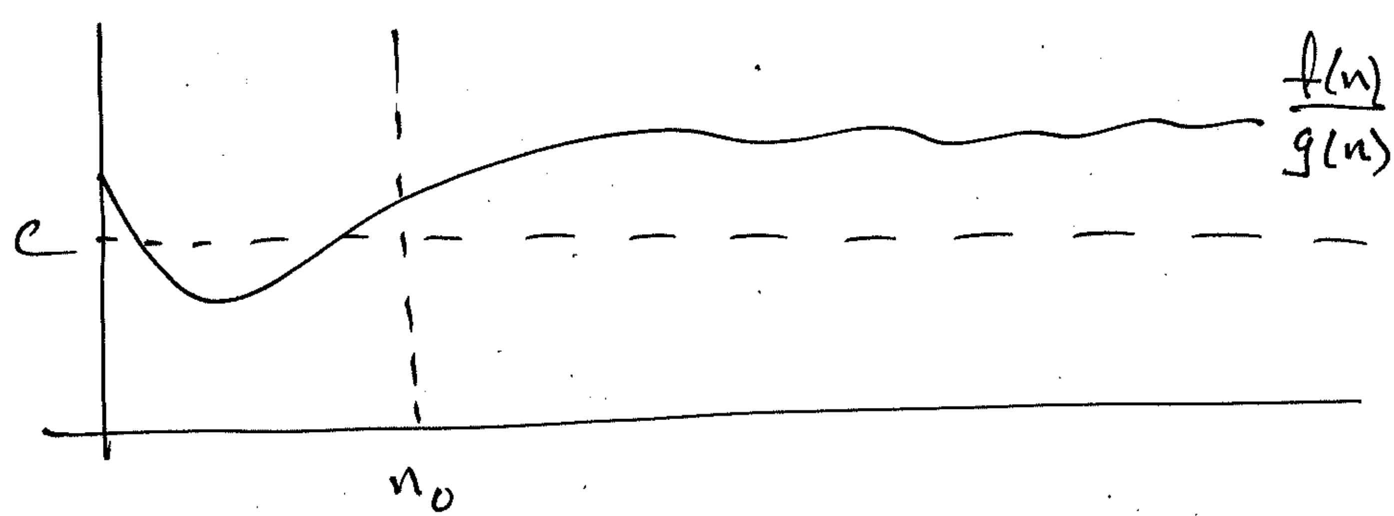
Defn let $f(n), g(n)$ be functions

$$\Omega(g(n)) = \{ f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c \cdot g(n) \leq f(n) \}$$

write : $f(n) = \Omega(g(n))$

note: if $g(n)$ is a.p., this is equivalent to

$$\exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c \leq \frac{f(n)}{g(n)}$$



Theorem

$$f(n) = O(g(n)) \text{ iff } g(n) = \Omega(f(n))$$

Proof ($\Rightarrow$)

Assume $f(n) = O(g(n))$. Then

(1) $\exists c_1 > 0, \exists n_1 > 0, \forall n \geq n_1 : 0 \leq f(n) \leq c_1 g(n)$

We must show $g(n) = \Omega(f(n))$, i.e.

(2) $\exists c_2 > 0, \exists n_2 > 0, \forall n \geq n_2 : 0 \leq c_2 f(n) \leq g(n)$

Let $c_2 = \dfrac{1}{c_1}$ and $n_2 = n_1$.

$\square$

Then $c_2 > 0$, and $n_2 > 0$, and

(1) implies (2). Hence

$$g(n) = \Omega(f(n)).$$

$\blacksquare$

Leave $(\Leftarrow)$ as exercise.

Defn $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$

In other words

$$f(n) = \Theta(g(n))$$

iff

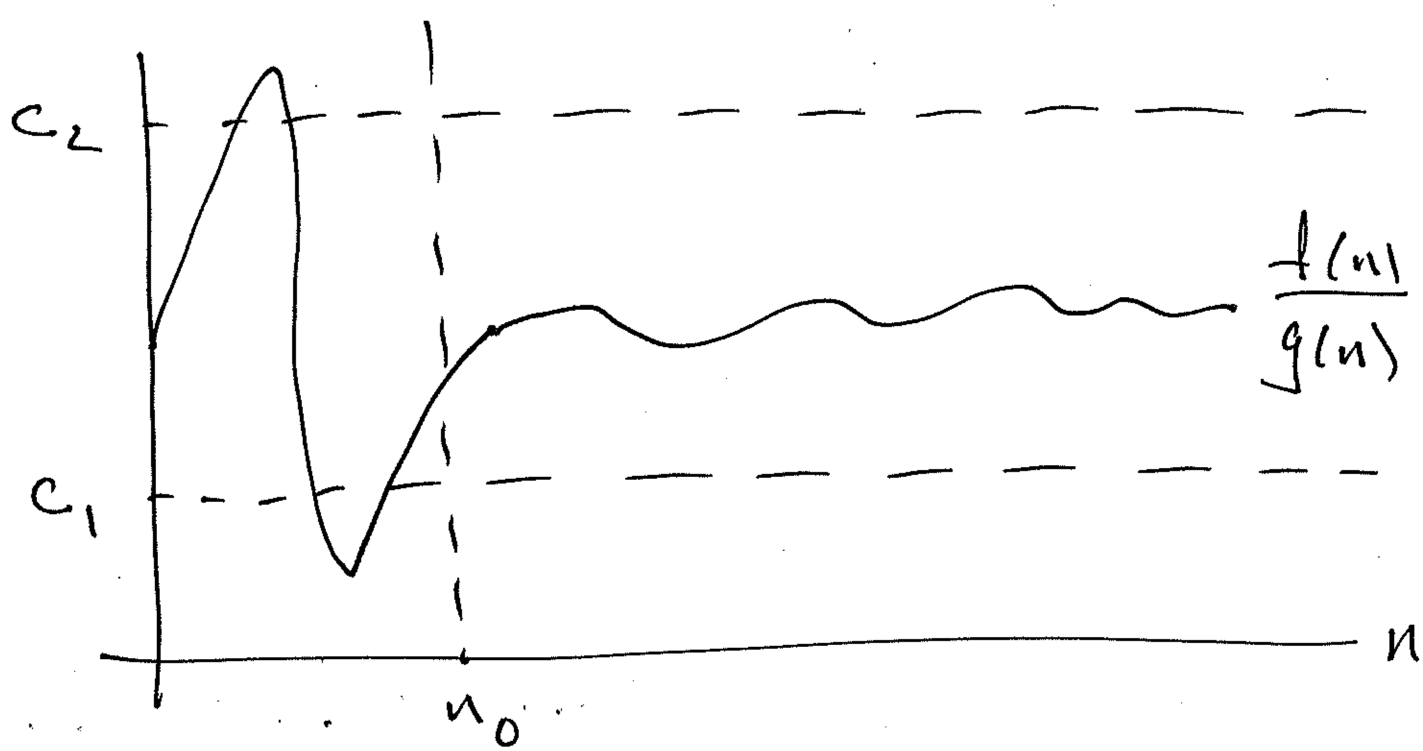
$$\exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \geq n_0 :$$

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

Note: if $g(n)$ is a.p. then this is equivalent to

$$\exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \geq n_0 :$$

$$0 \leq c_1 \leq \frac{f(n)}{g(n)} \leq c_2$$



Exercise

Prove $f(n) = \Theta(g(n))$ iff $g(n) = \Theta(f(n))$

Exercise

let $g(n)$ be a fcn, let $c > 0$.

Prove

$$c\,g(n) = O(g(n))$$

$$c\,g(n) = \Omega(g(n))$$

$$c\,g(n) = \Theta(g(n))$$