CSE 102 1-28-20

continue Recursion tree:

recurrence terminates when

$$\frac{n}{3^k} = 1$$

$$\therefore \quad 3^k = n$$

$$\therefore \quad k = \log_3 n$$

Value of nodes at level k : $T\left(\frac{n}{3^k}\right) = 1$

# of nodes at level $k = 2^k = 2^{\log_3 n} = n^{\log_3 2}$

Estimate:

$$T(n) = \sum_{i=0}^{k-1} 2^i \cdot \left(\frac{n}{3^i}\right) + 1 \cdot n^{\log_3 2}$$

$$\therefore\ T(n) = n\left(\sum_{i=0}^{k-1} \left(\tfrac{2}{3}\right)^i\right) + n^{\log_3 2}$$

$$= n\left(\frac{1-(2/3)^k}{1-(2/3)}\right) + n^{\log_3 2}$$

$$= 3n\left(1-\left(\tfrac{2}{3}\right)^k\right) + n^{\log_3 2}$$

Note: $T(n) \le 3n + n^{\log_3 2} = O(n)$

since $2 < 3$ we have $\log_3 2 < 1$.

Guess! $T(n) = O(n)$

<u>Exercise</u> use substitution method to Prove $T(n) = O(n)$, i.e. show $\forall n \ge n_0 : T(n) \le cn$

note: $T(n) = 2T(\frac{n}{3}) + n \geq n = \Omega(n)$

Hence: $T(n) = \Theta(n)$.

<u>Iteration Method:</u>

Same Problem

$$T(n) = \begin{cases} 1 & 1 \leq n < 3 \\ 2T(\lfloor \frac{n}{3} \rfloor) + n & n \geq 3 \end{cases}$$

$$T(n) = n + 2T(\lfloor \tfrac{n}{3} \rfloor)$$

$$= n + 2\left(\lfloor \tfrac{n}{3} \rfloor + 2T\left(\left\lfloor \tfrac{\lfloor n/3 \rfloor}{3} \right\rfloor\right)\right)$$

$$= n + 2\lfloor \tfrac{n}{3} \rfloor + 2^2 \cdot T\left(\lfloor \tfrac{n}{3^2} \rfloor\right)$$

$$= n + 2\lfloor \tfrac{n}{3} \rfloor + 2^2\left(\lfloor \tfrac{n}{3^2} \rfloor + 2T\left(\left\lfloor \tfrac{\lfloor n/3^2 \rfloor}{3} \right\rfloor\right)\right)$$

$$= n + 2\left\lfloor \frac{n}{3} \right\rfloor + 2^2\left\lfloor \frac{n}{3^2} \right\rfloor + 2^3 \cdot T\left(\left\lfloor \frac{n}{3^3} \right\rfloor\right)$$

$$\vdots$$

$$= \sum_{i=0}^{k-1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor + 2^k \cdot T\left(\left\lfloor \frac{n}{3^k} \right\rfloor\right)$$

Bottom out when : $1 \le \left\lfloor \frac{n}{3^k} \right\rfloor < 3$

i.e. $1 \le \frac{n}{3^k} < 3$

$\therefore \quad 3^k \le n < 3^{k+1}$

$\therefore \quad k \le \log_3 n < k+1$.

Hence $\boxed{k = \lfloor \log_3 n \rfloor}$.

$\therefore \quad 2^k = 2^{\lfloor \log_3 n \rfloor}$

$$\therefore\ T(n) = \sum_{i=0}^{k-1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor + 2^{\lfloor \log_3 n \rfloor}$$

upper bound :

$$T(n) \leq n \left(\sum_{i=0}^{k-1} \left(\frac{2}{3}\right)^i \right) + 2^{\log_3 n}$$

$$\leq n \left(\frac{1}{1-\frac{2}{3}} \right) + n^{\log_3 2}$$

$$= 3n + n^{\log_3 2} = O(n)$$

$$\therefore\ T(n) = O(n).$$

Also $T(n) = 2T(\lfloor \frac{n}{3} \rfloor) + n \geq n = \Omega(n)$

$$\therefore\ \boxed{T(n) = \Theta(n)}$$

Ex.

$$T(n) = \begin{cases} 5 & 0 \le n < 2 \\ T(n-2) + n & n \ge 2 \end{cases}$$

$$\begin{aligned} T(n) &= n + T(n-2) \\ &= n + (n-2) + T(n-2\cdot 2) \\ &= n + (n-2) + (n-2\cdot 2) + T(n-3\cdot 2) \\ &\vdots \\ &= \sum_{i=0}^{k-1} (n-2i) + T(n-2k) \end{aligned}$$

halts when: $0 \le n - 2k < 2$

$\therefore\ 2k \le n < 2 + 2k$

$\therefore\ k \le \frac{n}{2} < k+1$

$\square$

$\therefore$ $\boxed{K = \lfloor \frac{n}{2} \rfloor}$

Thus

$$T(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} (n - 2i) + 5$$

$$= n \cdot \sum_{i=0}^{K-1} 1 - 2 \sum_{i=\cancel{0},1}^{K-1} i + 5$$

$$= n \cdot K - \cancel{2} \frac{K(K-1)}{\cancel{2}} + 5$$

$$\boxed{T(n) = n \cdot \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor \left(\lfloor \frac{n}{2} \rfloor - 1 \right) + 5}$$

By earlier results:

$$\boxed{T(n) = \Theta(n^2)}$$.

Master Theorem

Define $T(n)$ by

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where $a \geq 1$, $b > 1$, $f(n)$ is a.p.

we have 3 cases

(1) if $f(n) = O(n^{\log_b a - \varepsilon})$ for some $\varepsilon > 0$

then

$$\boxed{T(n) = \Theta(n^{\log_b a})}$$

(2) if $f(n) = \Theta(n^{\log_b a})$, then

$$\boxed{\begin{aligned} T(n) &= \Theta(n^{\log_b a} \cdot \log n) \\ &= \Theta(f(n) \log n) \end{aligned}}$$

(3) if $f(n) = \Omega(n^{\log_b a + \varepsilon})$, and

if $a f(\frac{n}{b}) \le c f(n)$ for some $c \in (0, 1)$, and all suff. large n, then

regularity cond.

$$\boxed{T(n) = \Theta(f(n))}$$

Remarks

- Compare $f(n)$ to $n^{\log_b a}$

- If $f(n) = \Theta(n^d)$, then easy to compare.

  (1) $d < \log_b a$ let $\varepsilon = \log_b a - d$

  (2) $d = \log_b a$

  (3) $d > \log_b a$. let $\varepsilon = d - \log_b a$

Exercise

Show that if $f(n) = \Theta(n^d)$ where $d > \log_b a$, then regularity cond. is satisfied.

Exercise

a.p.

find a function $f(n)$ such that $f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right)$ for some $a \geq 1$, $b > 1$, $\varepsilon > 0$), But regularity cond. fails.

Ex. $T(n) = T\left(\lfloor \frac{n}{2} \rfloor\right) + 2T\left(\lceil \frac{n}{2} \rceil\right) + \log(n!)$

simplify:

$$T(n) = 3T\left(\frac{n}{2}\right) + n\log n$$

compare: $n\log n$ to $\underbrace{n^{\log_2 3}}_{\text{winner}}$

let $\varepsilon = \frac{1}{2}(\log_2 3 - 1) > 0$

$2\varepsilon = \log_2 3 - 1$

$\varepsilon + 1 = \log_2 3 - \varepsilon$

$$\frac{n\log n}{n^{\log_2 3 - \varepsilon}} = \frac{n\log n}{n^{1+\varepsilon}} = \frac{\log n}{n^{\varepsilon}} \to 0 \text{ as } n \to \infty$$

$\therefore\ n\log n = o(n^{\log_2 3 - \varepsilon}) \subseteq O(n^{\log_2 3 - \varepsilon})$

By case (1):

$$\boxed{T(n) = \Theta(n^{\log_2 3})}$$

check $\varepsilon = \log_2 3 - 1$ does not work.

$$\frac{n \log n}{n^{\log_2 3 - \varepsilon}} = \frac{\cancel{n} \log n}{\cancel{n}} \longrightarrow \infty$$

$\therefore$ $n \log n = \omega(n^{\log_2 3 - \varepsilon})$

also $\omega(n^{\log_2 3 - \varepsilon}) \cap O(n^{\log_2 3 - \varepsilon}) = \phi$

$\therefore$ $n \log n \neq O(n^{\log_2 3 - \varepsilon})$.

Divide & Conquer algorithms

<u>Example</u>
MergeSort

A[p .... r] unsorted

A[p .... q]   A[q ..... r] "

sort   sort

A[p .... q]   A[q .... r] sorted

merge

A[p .... r] sorted

MergeSort(A, p, r)

```
if p < r
    q = ⌊(p+r)/2⌋
    MergeSort(A, p, q)
    MergeSort(A, q+1, r)
    Merge(A, p, q, r)
```

found in sec. 2.3 p. 31

Theorem

Assuming correctness of Merge( ), After MergeSort(A, p, r), the subarray A[p ... r] is sorted.

Proof

Induction on $m = \text{length}(A[p \ldots r])$
$= r - p + 1$

I. if $m = 1$, then $r - p + 1 = 1$ and $r = p$, so M.S. does nothing. Indeed $A[p \ldots r]$ has just 1 element, so is sorted.

II. Let $m > 1$. Assume M.S. is correct on any subarray of length $1 \le k < m$.

must show correct on length m.

$m > 1$ implies $r > p$ so line 2 sets $q = \lfloor \frac{r+p}{2} \rfloor$

Exercise: check $p \le q < r$

$\therefore$ Length$(A[p \dots q]) = q - p + 1 < r - p + 1 = m$

$\therefore$ Length$(A[q+1 \dots r]) = r - (q+\cancel{1}) + \cancel{1} < r - p + 1 = m$

$\therefore$ By ind hyp., Rec. calls on lines 3 & 4 correctly sort subarrays

$A[p \ldots q]$

and

$A[q+1 \ldots r]$

correctness of Merge(A, p, q, r) implies that $A[p \ldots r]$ is sorted at end. ∎