CSE 102 1-23-20

Substitution method

Ex

$$T(n) = \begin{cases} 2 & 1 \le n < 3 \\ 3T(\lfloor \frac{n}{3} \rfloor) + n & n \ge 3 \end{cases}$$

Guess: $T(n) = O(n \log n)$

To prove, show $\exists\, c, n_0$ s.t.

$\forall n \ge n_0$: $T(n) \le c n \log(n)$

mimic ind. Proof.

$T(1) = 2$ but $c \cdot 1 \cdot \log 1 = 0$
and $2 \not\leq 0$, so $n=1$ fails

$T(2) \leq c\, 2 \log 2$ says $2 \leq 2 c \log 2$
which says $1 \leq c \log 2$, can
be made true, by choice of c.

ind. step.

$\begin{cases} n_0 = 2 \text{ lowest} \\ n_1 = \text{(unknown) highest} \end{cases}$

let $n > n_1 \leftarrow$ highest base case
Assume: $T(k) \leq c k \log k \quad (n_0 \leq k < n)$
show: $T(n) \leq c n \log n$.

SO

$$T(n) = 3T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n$$

$$\leq 3 \cdot c \left\lfloor \frac{n}{3} \right\rfloor \log \left\lfloor \frac{n}{3} \right\rfloor + n$$

$\left\{ \begin{array}{l} \text{by ind. hyp. with} \\ k = \left\lfloor \frac{n}{3} \right\rfloor \text{. need} \\ n_0 \leq \left\lfloor \frac{n}{3} \right\rfloor < n \end{array} \right.$

$$\leq \cancel{3} c \left(\frac{n}{\cancel{3}}\right) \log\left(\frac{n}{3}\right) + n \quad \left\{ \begin{array}{l} \text{since} \\ \lfloor x \rfloor \leq x \end{array} \right.$$

$$= cn\left(\log(n) - \log(3)\right) + n \quad \left\{ \boxed{\begin{array}{l} \text{Pick} \\ \log(\cdot) = \log_3(\ ) \end{array}} \right.$$

$$= cn\log n - cn + n \underset{\substack{\uparrow \\ \text{want}}}{\leq} cn\log n$$

i.e. $\quad -cn + n \leq 0$

$$n \leq cn$$

$$1 \leq c$$

Also need $n_0 \le \lfloor \frac{n}{3} \rfloor < n$ (for $n \geq n_1$)

Try $\boxed{n_0 = 2}$, so need

$$2 \le \left\lfloor \frac{n}{3} \right\rfloor < n$$

$$\therefore \quad 2 \le \frac{n}{3}$$

$$\therefore \quad 6 \le n \quad \text{true if } \boxed{n_1 = 5}$$

Base cases

$n = 2$

$n = 3$: $T(3) \le c \cdot 3 \log_3(3)$

$$9 \le 3c$$

$$c \geq 3$$

$n=4$: $T(4) \le c \cdot 4 \log_3(4)$

$$10 \le 4c \log_3(4)$$

$$c \ge \frac{5}{2\log_3(4)}$$

$n=5$: $T(5) \le c \cdot 5 \log_3(5)$

$$11 \le 5c \log_3(5)$$

$$c \ge \frac{11}{5\log_3(5)}$$

check

$$3 > \frac{5}{2\log_3(4)} > \frac{11}{5\log_3(5)}$$

Pick $\boxed{c = 3}$

Now prove if

$$T(n) = \begin{cases} 2 & 1 \le n < 3 \\ 3T(\lfloor \frac{n}{3} \rfloor) + n & n \ge 3 \end{cases}$$

Then for all $n \ge 2$: $\boxed{T(n) \le 3n\log_3(n)}$

(hence : $T(n) = O(n \log n)$.) $P(n)$

Proof

I. we have base cases

$n = 2$ ✓

$n = 3$: $T(3) \le 3 \cdot 3 \log_3(3)$

$9 \le 9 \cdot 1$ ✓

$n = 4$: $T(4) \le 3 \cdot 4 \log_3(4)$

$10 \le 12 \log_3(4)$ ✓

$n=5$: $T(5) \le 3 \cdot 5 \log_3(5)$

$11 \le 15 \log_3(5)$ ✓

II d. $\forall n > 5 : P(2) \wedge \ldots \wedge P(n-1) \rightarrow P(n)$

Let $n > 5$ be arbitrary.
Assume for all k in range
$2 \le k < n$ that

$$T(k) \le 3k \log_3(k).$$

must show that

$$T(n) \le 3n \log_3(n).$$

So

$$T(n) = 3T(\lfloor \tfrac{n}{3} \rfloor) + n$$

$$\le 3 \cdot 3 \lfloor \tfrac{n}{3} \rfloor \log_3 \lfloor \tfrac{n}{3} \rfloor + n$$

by ind. hyp. with $k = \lfloor \frac{n}{3} \rfloor$. Poss. since $2 \le \lfloor \frac{n}{3} \rfloor < n$

$$\le 9\left(\tfrac{n}{3}\right) \log_3\left(\tfrac{n}{3}\right) + n$$

since $\lfloor x \rfloor \le x$

$$= 3n(\log_3 n - 1) + n$$

$$= 3n\log_3 n - 3n + n$$

$$= 3n\log_3 n - 2n \le 3n\log_3 n.$$

Result follows by 2^{nd} PMI.

$\blacksquare$

Ex.

$$T(n) = \begin{cases} 1 & n=1 \\ 2T(\lfloor \frac{n}{2} \rfloor) + 1 & n \geq 2. \end{cases}$$

Guess : $T(n) = O(n)$

find Pos. c, n_0 s.t.

$$\forall n \geq n_0 : \boxed{T(n) \leq cn}$$

mimic ind. step.

let $n > n_0$.

Assume: $T(k) \leq ck$ for $(n_0 \leq k < n)$

Show : $T(n) \leq cn$

So

$$T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1$$

$$\le 2c\left\lfloor \frac{n}{2} \right\rfloor + 1$$ by ind. hyp. $k = \left\lfloor \frac{n}{2} \right\rfloor$. $n_0 \le \left\lfloor \frac{n}{2} \right\rfloor < n$

$$\le \cancel{2}c \cdot \left(\frac{n}{\cancel{2}}\right) + 1$$

$$= cn + 1 \le cn$$

want

i.e. $1 \le 0$ ✗

Trick: Subtract a lower order term from cn

show: $\boxed{T(n) \le cn - b}$ for all $n \ge n_0$. $(c, b \in \mathbb{R}^+)$

mimic induction step

Let $n > n_0$

Assume: $T(k) \le ck - b \quad (n_0 \le k < n)$

Show: $T(n) \le cn - b$

$$T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1$$

$$\le 2\left(c\left\lfloor \frac{n}{2} \right\rfloor - b\right) + 1 \qquad \left\{ \begin{array}{l} \text{by ind. hyp} \\ k = \left\lfloor \frac{n}{2} \right\rfloor \end{array} \right.$$

$$\le \cancel{2}c\left(\frac{n}{\cancel{2}}\right) - 2b + 1 \qquad \left\{ \lfloor x \rfloor \le x \right.$$

$$= cn - 2b + 1 \underset{\substack{\uparrow \\ \text{want}}}{\le} cn - b$$

i.e. $-2b + 1 \le -b$

$$1 \le b$$

Base case: $T(n_0) \le c n_0 - b$

Try $n_0 = 1$: $1 \le c - b$

$$c \ge 1 + b$$

Pick $\boxed{b=1}$ and $\boxed{c=2}$, $\boxed{n_0=1}$

<u>Exercise</u>: show

$$\forall n \ge 1: \; T(n) \le 2n - 1$$

Recursion Tree / Iteration

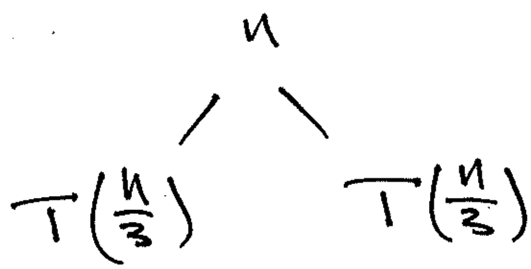
Ex.

$$T(n) = \begin{cases} 1 & 1 \le n < 3 \\ 2T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n & n \ge 3 \end{cases}$$

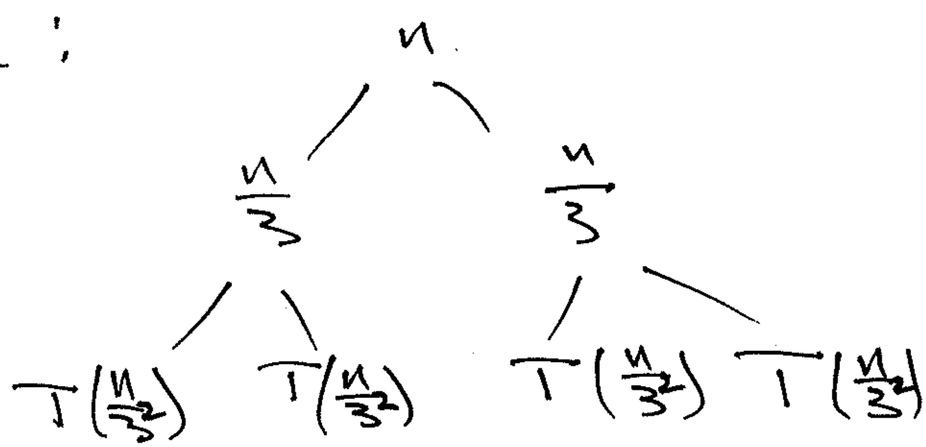
write: $\boxed{T(n) = n + T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + T\left(\left\lfloor \frac{n}{3} \right\rfloor\right)}$

0th tree: $T(n)$

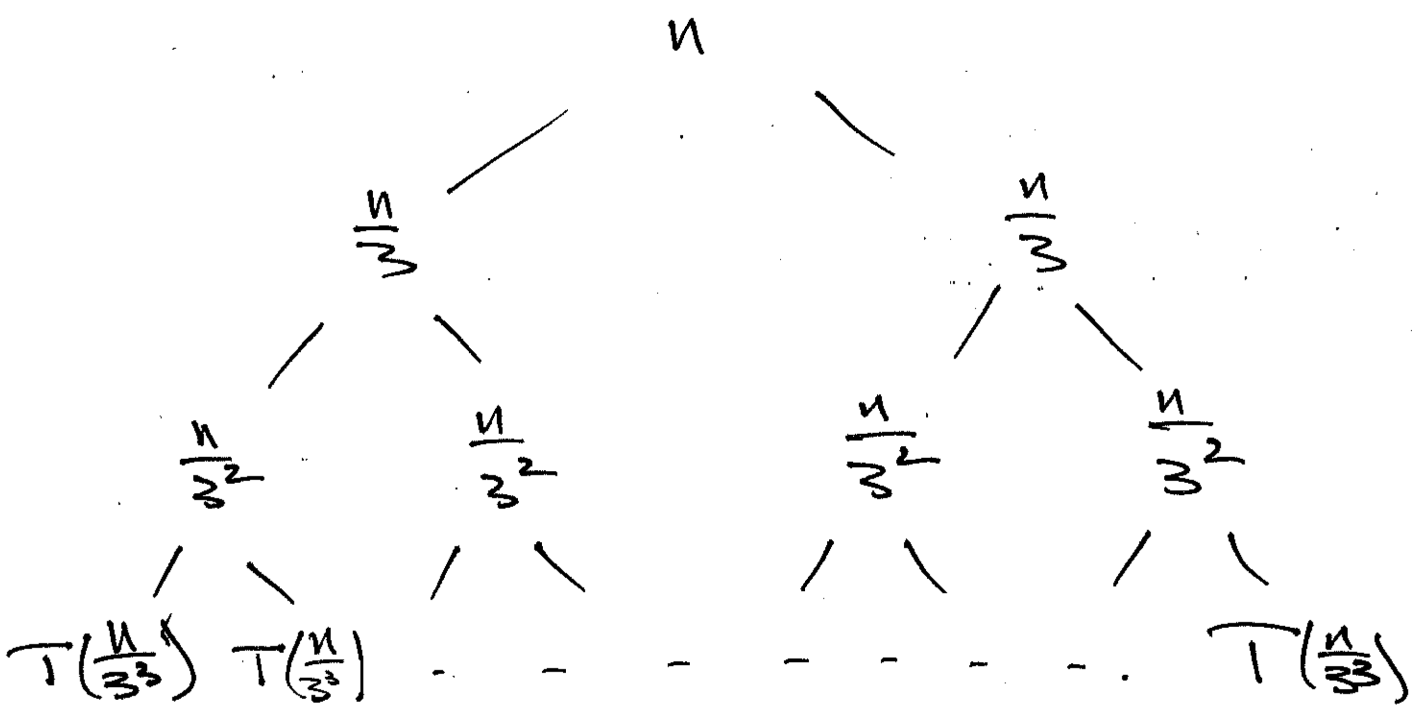
1st tree:



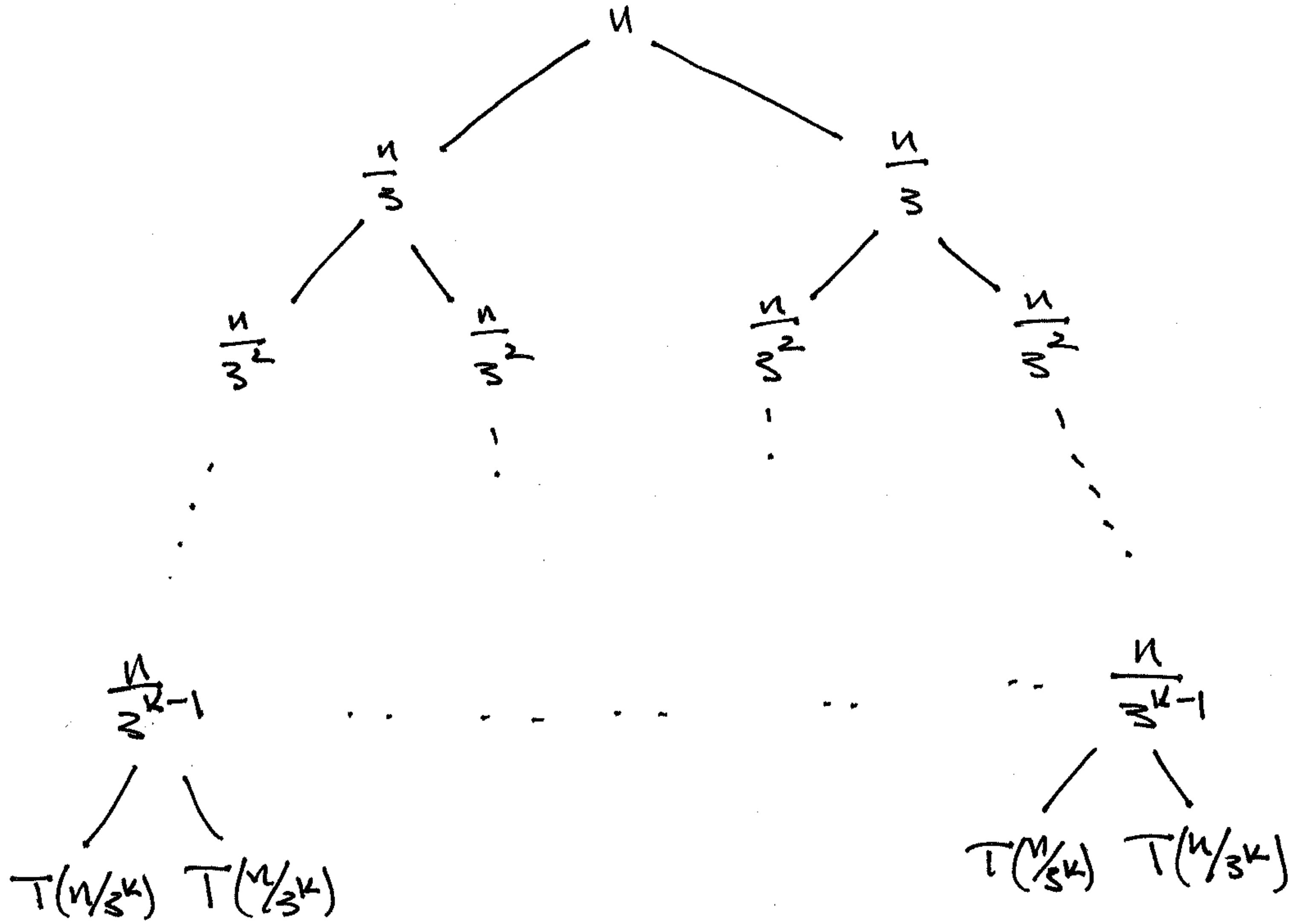
2nd tree:



3rd tree:



k^{th} tree:



nodes at depth i: 2^i $(0 \le i \le k)$

Value of nodes at depth i $(0 \le i \le k-1)$: $\frac{n}{3^i}$

Value of nodes at depth k: $T\left(\frac{n}{3^k}\right)$