CSE 102 1-21-20

continue....

II d. $\forall n \geq 2: P(1) \wedge \ldots \wedge P(n-1) \rightarrow P(n)$

let $n \geq 2$ be arbitrary. Assume for any k in range $1 \leq k < n$ that

$$T(k) \leq k^2.$$

we must show: $T(n) \leq n^2$.

so

$$T(n) = 4T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n \quad \{\text{by rec. relation}$$

$$\leq 4 \cdot \left\lfloor \frac{n}{3} \right\rfloor^2 + n \quad \begin{cases} \text{by ind. hyp. and} \\ 1 \leq \left\lfloor \frac{n}{3} \right\rfloor < n \end{cases}$$

$\le 4\left(\frac{n}{3}\right)^2 + n$ $\quad$ {since $\lfloor x \rfloor \le x$

$= \frac{4}{9}n^2 + n \le n^2$

want. true since

$n > 2 \to \frac{9}{5} \le n \to n \le \frac{5}{9}n^2 \to \frac{4}{9}n^2 + n \le n^2$

$\therefore T(n) \le n^2$ for all $n \ge 1$ by

2nd PMI. ■

note

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...
$\lfloor \frac{n}{3} \rfloor$	0	0	1	1	1	2	2	2	3	3	3	4	4	4	..

Example C : induction fallacy

show : All horses are of the same color.

show $\forall n \geq 1$: $\underbrace{\boxed{\text{if } S \text{ is a set of } n \text{ horses, then all horses in } S \text{ have same color}}}_{P(n)}$

Proof (invalid !!)

I. P(1) says "in any set consisting of 1 horse, All horses have same color.

IIa. $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$

Let $n \geq 1$ be arbitrarily. Assume:

if T is a set of n horses, then, all horses in T have same color

we must show:

if S is a set of n+1 horses, all horses in S have same color.

Let S be any set of n+1 horses.

Say

$$S = \{h_1, h_2, h_3, \ldots, h_{n+1}\}.$$

Define

$$S' = S - \{h_1\}$$

$$S'' = S - \{h_2\}$$

Since both S' and S'' are sets of n horses, all horses in S' have same color, and likewise for S''. Notice

$$h_3 \in S' \cap S''$$

$(n > 1 \rightarrow n \geq 2 \rightarrow n+1 \geq 3$, so h_3 exists.)

so all horses in S' have same color as h_3, as do all horses in S'', so all horses in

$$S' \cup S'' = S$$

have same color. $\blacksquare$

Why is Induction Valid?

1st Principle of Mathematical Ind.

for any Prop. fcn. $P(n)$:

$$\Big(P(1) \wedge \big[\forall n>1 : P(n-1) \rightarrow P(n)\big]\Big) \rightarrow \forall n \geq 1 : P(n)$$

2nd PMI

for any Prop. fcn $Q(n)$:

$$\Big(Q(1) \wedge \big[\forall n>1 : Q(1) \wedge \cdots \wedge Q(n-1) \rightarrow Q(n)\big]\Big) \rightarrow \forall n \geq 1 : Q(n)$$

Well ordering Property of $\mathbb{Z}^+$ (WOP)

Any non-empty subset of $\mathbb{Z}^+$ contains a least element.

Theorem

1^{st} PMI, 2^{nd} PMI, and WOP are logically equivalent.

$$1^{st}\text{ PMI} \Leftrightarrow 2^{nd}\text{ PMI} \Leftrightarrow \text{WOP}.$$

HW#6: 1^{st} PMI $\Rightarrow$ 2^{nd} PMI.

we prove that

$$\text{WOP} \Rightarrow 1^{st} \text{ PMI}$$

<u>Proof</u>. Assume WOP holds. Let $P(n)$ be any propositional function. Assume

(1) $P(1)$

and

(2) $\forall n > 1 : P(n-1) \Rightarrow P(n)$

we must show

(3) $\forall n \geq 1 : P(n)$

Let $S = \{n \geq 1 : P(n) \text{ is false}\}$.

It is suff. to show $S = \emptyset$, for then (3) is true.

Assume, to get a ※, that $S \neq \emptyset$. Then S is a non-empty subset of $\mathbb{Z}^+$. By the WOP, S contains a least element, call it m. Thus

$$m \in S \text{ and } k < m \Rightarrow k \notin S.$$

By (1), $P(1)$ is true, so $1 \notin S$ so $m \neq 1$. $\therefore m > 1$, and hence $m-1 \geq 1$. Since $m-1 < m$ we have $m-1 \notin S$, so $\boxed{P(m-1) \text{ is true}}$. By (2) with $n = m$, we have

$\boxed{P(m-1) \rightarrow P(m) \text{ is true}}$.

Hence $P(m)$ is also true by Modus Ponens, and therefore $m \notin S$. This contradicts that $m \in S$.

This $\times$ shows our

our assumption was false, hence $S = \emptyset$ and (3) holds. $\blacksquare$

<u>hint on hw#6</u>: 1^{st} PMI $\Rightarrow 2^{nd}$ PMI.

Assume 1^{st} PMI: ... $P(n)$ ...

Let $Q(n)$ be any prop. fcn.

Assume

(1) $Q(1)$

and

(2) $\forall n > 1 : Q(1) \wedge \dots \wedge Q(n-1) \Rightarrow Q(n)$

Show

(3) $\forall n \geq 1 : Q(n)$

Define

$P(n) = \ldots\ldots\ldots Q(\cdot) \ldots\ldots\ldots$

If you choose $P(n)$ correctly,

$P(1) = Q(1)$

$P(2) = ?$

$P(3) = ?$

$\vdots$

$P(n) = ?$

Recurrence Relations

Goal: analyze recurrences of the form

$$T(n) = \begin{cases} c & n=1 \\ T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n}{2} \right\rceil\right) + n \cdot d & n \geq 1. \end{cases}$$

3 methods

- Substitution
- Recursion-tree / Iteration
- Master Theorem.