cse 102 1-16-20

Recall Stirling's formula

$$n! = \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)$$

Corollary

(1) $n! = o(n^n)$

(2) $n! = \omega(b^n)$ any $b > 0$ (Hint: $b^n = e^{n \cdot \ln b}$)

(3) $\log(n!) = \Theta(n \log n)$

Proof of (1)

$$\frac{n!}{n^n} = \frac{\sqrt{2\pi n} \cdot \frac{\cancel{n^n}}{e^n} \cdot \left(1+\Theta\left(\frac{1}{n}\right)\right)}{\cancel{n^n}}$$

$$= \frac{\sqrt{2\pi} \cdot n^{1/2} \cdot \left(1+\Theta\left(\frac{1}{n}\right)\right)}{e^n} \longrightarrow 0$$

as $n \to \infty$

■

Exercise: Prove (2)

Proof of (3)

$$\log(n!) = \log\sqrt{2\pi} + \log\sqrt{n} + \log n^n - \log e^n + \log(\qquad)$$

$$= \log\sqrt{2\pi} + \frac{1}{2}\log n + n\log n - n\log e + \log(\qquad)$$

>0

$$\underbrace{\frac{\log(n!)}{n\log n}}_{\downarrow \atop 1 \text{ as } n\to\infty} = \underbrace{\frac{\log\sqrt{2\pi}}{n\log n}}_{\downarrow \atop 0} + \underbrace{\frac{1}{2n}}_{\downarrow \atop 0} + \underbrace{1}_{\downarrow \atop 1} - \underbrace{\frac{\log e}{\log n}}_{\downarrow \atop 0} + \underbrace{\frac{\log(\qquad)}{n\log n}}_{\downarrow \atop 0}$$

$\blacksquare$

<u>Ex</u> $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$ ✓

<u>Recall</u> $\binom{m}{k} = \frac{m!}{k!\,(m-k)!}$

$\binom{m}{k} \overset{\text{defn.}}{=}$ # of k-subsets of an m-set. $(0 \le k \le m)$

Pascal's Δ

n	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
$\vdots$	$\vdots$

Proof

$$\binom{2n}{n} = \frac{(2n)!}{n!\,(2n-n)!}$$

$$= \frac{(2n)!}{(n!)^2}$$

$$= \frac{\sqrt{2\pi\cdot 2n}\cdot\left(\frac{2n}{e}\right)^{2n}\cdot\left(1+\Theta\left(\frac{1}{2n}\right)\right)}{\left(\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^{n}\left(1+\Theta\left(\frac{1}{n}\right)\right)\right)^{2}}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{n}}\cdot\frac{\frac{2^{2n}\cdot\cancel{n^{2n}}}{\cancel{e^{2n}}}}{\frac{\cancel{n^{2n}}}{\cancel{e^{2n}}}}\cdot\left[\frac{\left(1+\Theta\left(\frac{1}{2n}\right)\right)}{\left(1+\Theta\left(\frac{1}{n}\right)\right)^{2}}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{4^{n}}{\sqrt{n}}\cdot\left[\qquad\right]$$

> 0

$$\frac{\binom{2n}{n}}{\frac{4^{n}}{\sqrt{n}}} = \frac{1}{\sqrt{2\pi}}\cdot\left[\qquad\right] \xrightarrow[\text{as } n\to\infty]{} \frac{1}{\sqrt{2\pi}} \in (0,\infty)$$

$\blacksquare$

Exercise

find const $a > 0$ s.t.

$$\binom{3n}{n} = \Theta\left(\frac{a^n}{\sqrt{n}}\right)$$

In general find $a_k > 0$ s.t.

$$\binom{kn}{n} = \Theta\left(\frac{a_k^n}{\sqrt{n}}\right).$$

Induction Proofs

Let $P(n)$ be a propositional function,

$$P : \mathbb{Z}^+ \longrightarrow \{\text{Propositions depending on } n\}$$

↑
or another subset of $\mathbb{Z}$.

Effectively $P : \mathbb{Z}^+ \longrightarrow \{\text{true}, \text{false}\}$

Goal: Prove $\forall n \geq 1 : P(n)$

I. base case: show $P(1)$

IIa. induction step

show $\forall n \geq 1 : \underbrace{P(n)}_{\text{induction hypothesis}} \to P(n+1)$

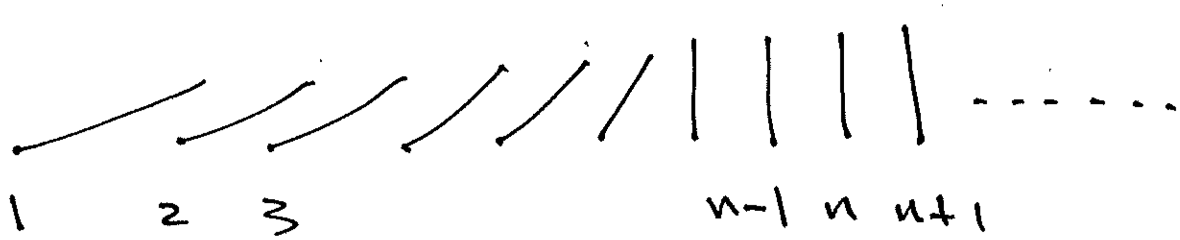
conclusion: $\forall n \geq 1 : P(n)$.

like an infinite "proof":

1.	$P(1)$	I
2.	$P(1) \rightarrow P(2)$	II
3.	$P(2)$	modus ponens
4.	$P(2) \rightarrow P(3)$	II
5.	$P(3)$	M.P.

$\vdots$

Domino analogy



$P(n) =$ "the n^{th} domino falls"

I : $P(1) =$ " the 1^{st} domino falls"

IIa. $\forall n \geq 1 : P(n) \rightarrow P(n+1)$

= "if any domino falls, then so does the next"

Conclusion: $\forall n \geq 1 : P(n)$

= "all dominos fall"

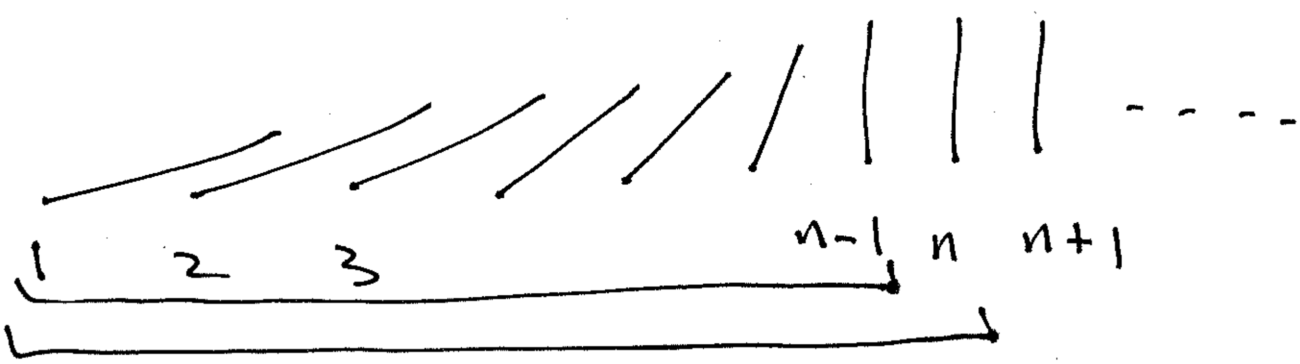
<u>Variations</u> :

IIb. $\forall n > 1 : \underbrace{P(n-1)}_{\text{ind. hyp.}} \rightarrow P(n)$

II a & II b are called

- weak induction
- 1^{st} Principle of Mathematical induction (PMI).

II c. $$\forall n \geq 1 : \underbrace{P(1) \wedge P(2) \wedge \ldots \wedge P(n)}_{\underbrace{\forall k : 1 \leq k \leq n : P(k)}_{\text{induction hyp.}}} \rightarrow P(n+1)$$



$$\text{II}d.\quad \forall n>1:\ \underbrace{\underbrace{P(1)\wedge\cdots\wedge P(n-1)}_{\forall k:\,1\le k<n\,:\,P(k)}}_{\text{ind. hyp.}} \longrightarrow P(n)$$

IIc & IId are called

- <u>strong</u> induction
- 2^{nd} PMI

<u>Ex.</u> define

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1 & \text{if } n \geq 2 \end{cases}$$

show: $\forall n \geq 1 : \boxed{T(n) \leq \lg(n)}$ $\leftarrow P(n)$

(hence $T(n) = O(\log n)$)

<u>Proof.</u>

I. $P(1)$ ✓

show $T(1) \leq \lg(1)$ ✓

i.e. $0 \leq 0$ ✓

II d. $\forall n > 1 : P(1) \wedge \dots \wedge P(n-1) \rightarrow P(n)$

$\forall k : 1 \le k < n : P(k)$

- Let $n > 1$ be chosen arbitrarily.
- Assume for all k in the range $1 \le k < n$ that $T(k) \le \lg(k)$.
- we must show that $T(n) \le \lg(n)$.

Then

$T(n) = T(\lfloor \frac{n}{2} \rfloor) + 1$ by rec. for $T(n)$

$\le \lg(\lfloor \frac{n}{2} \rfloor) + 1$ $\left\{ \begin{array}{l} \text{by ind. hyp.} \\ \text{with } k = \lfloor \frac{n}{2} \rfloor \end{array} \right.$

$\le \lg(\frac{n}{2}) + 1$ since $\lfloor x \rfloor \le x$

$= \lg n - \cancel{1} + \cancel{1}$

$= \lg n$

Hence $T(n) \le \lg(n)$ for all $n \ge 1$ by the 2nd PMI. ∎

Multiple base cases:

I. Prove $P(1), P(2), \ldots, P(n_0)$

II. $\forall n > n_0 : P(1) \wedge \ldots \wedge P(n-1) \rightarrow P(n)$

Conclusion: $\forall n \ge 1 : P(n)$

Ex. Define

$$T(n) = \begin{cases} 1 & \text{if } n=1 \text{ or } n=2 \\ 4T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n & \text{if } n \geq 3 \end{cases}$$

show $\forall n \geq 1$: $\boxed{T(n) \leq n^2}$ ← $P(n)$

(hence $T(n) = O(n^2)$)

Proof.

I. $P(1)$ says $T(1) \leq 1^2$, i.e.

$1 \leq 1^2$, i.e. $1 \leq 1$. ✓

$P(2)$ says $T(2) \leq 2^2$, i.e.

$1 \leq 4$, which is true. ✓

II d. $\forall n > 2 : \underbrace{P(1) \wedge \ldots \wedge P(n-1)}_{\forall k : 1 \le k < n : P(k)} \to P(n)$

Let $n > 2$ be arbitrary. Then

$$T(n) = 4\,T\left(\left\lfloor \tfrac{n}{3} \right\rfloor\right) + n$$

$$\le 4 \cdot \left\lfloor \tfrac{n}{3} \right\rfloor^2 + n \quad \left\{ \begin{array}{l} \text{by ind. hyp.} \\ \text{with } k = \left\lfloor \tfrac{n}{3} \right\rfloor \\ \text{since } 1 \le \left\lfloor \tfrac{n}{3} \right\rfloor < n \end{array} \right.$$

$\vdots$