Theorem

If $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = L$, where $0 \le L < \infty$,

then $f(n) = O(g(n))$.

Rmk. converse is false!

Proof

The defn. of the limit says

$$\forall \varepsilon > 0,\ \exists n_0 > 0,\ \forall n \ge n_0 : \left|\frac{f(n)}{g(n)} - L\right| < \varepsilon$$

Let $\varepsilon = 1$. Then

$$\exists n_0 > 0, \forall n \geq n_0 : \left| \frac{f(n)}{g(n)} - L \right| < 1$$

$$'' \qquad '' \qquad -1 < \frac{f(n)}{g(n)} - L < 1$$

$$'' \qquad '' \qquad \frac{f(n)}{g(n)} < (1 + L)$$

$$\therefore \quad 0 \leq f(n) \leq (1+L) g(n) .$$

Let $c = 1 + L$ in defn. of $O(g(n))$

note $c > 0$ since $L \geq 0$. ∎

Theorem

If $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = L$, where $0 < L \le \infty$,

then $f(n) = \Omega(g(n))$. { note: converse false!

Proof.

Let $L' = \frac{1}{L}$. Then $0 \le L' < \infty$

and

$$\lim_{n\to\infty}\left(\frac{g(n)}{f(n)}\right) = L'.$$

By last Thm: $g(n) = O(f(n))$.

By another Thm: $f(n) = \Omega(g(n))$.

$\blacksquare$

Theorem

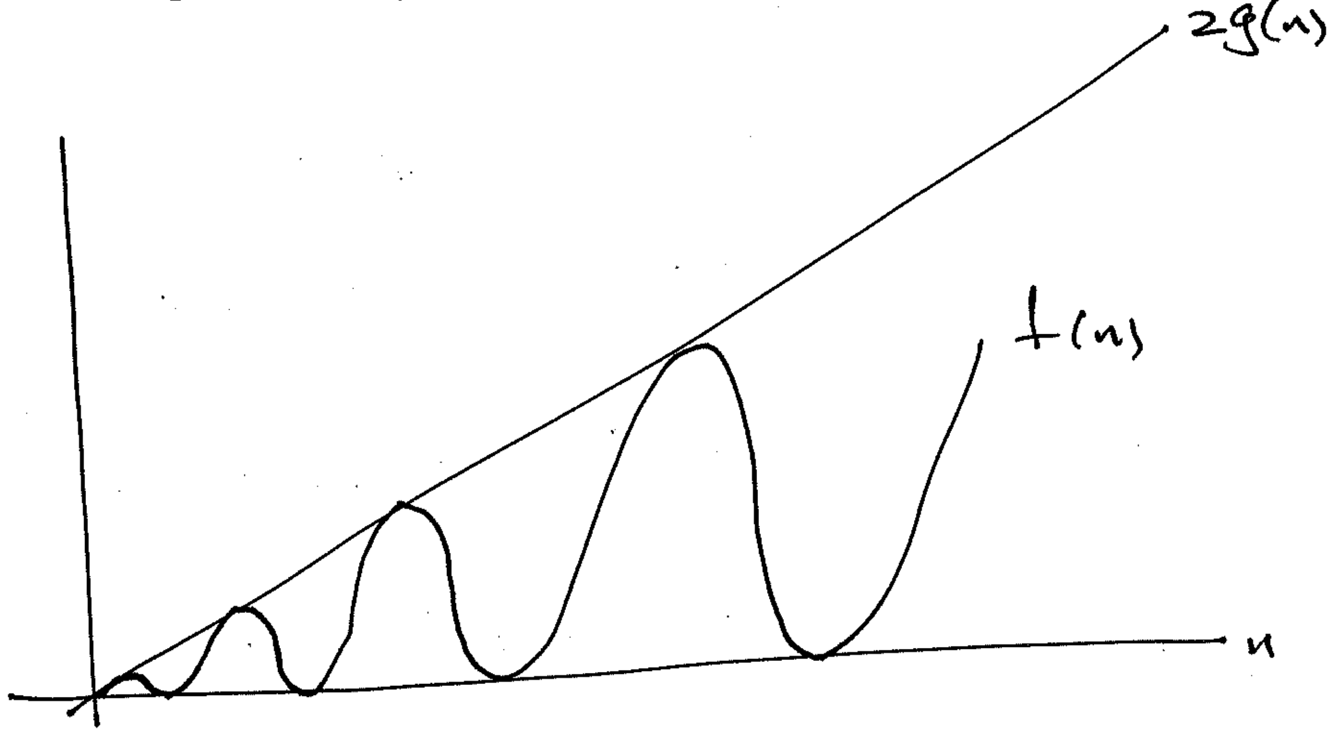
If $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = L$, where $0 < L < \infty$,

then $f(n) = \Theta(g(n))$.

note: converse is false!

Ex let $g(n) = n$, $f(n) = (1 + \sin(n))n$

(A)

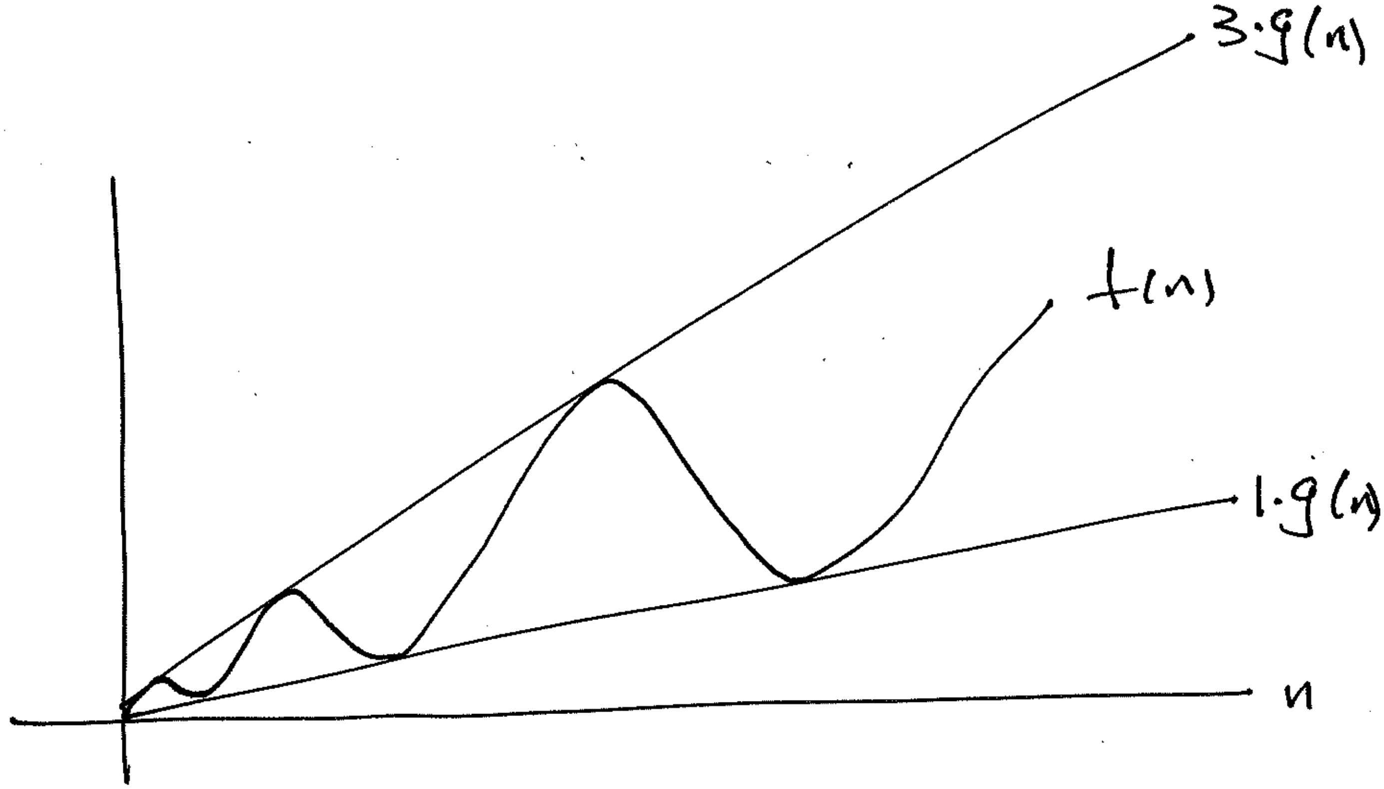


so $f(n) = O(g(n))$, but $f(n) \neq \Omega(g(n))$

Also

$$\frac{f(n)}{g(n)} = 1 + \sin(n), \text{ limit D.N.E.}$$

Ex. (B) let $g(n) = n$, $f(n) = (2 + \sin(n))\, n$



$\therefore\ f(n) = \Theta(g(n))$

But $\dfrac{f(n)}{g(n)} = 2 + \sin(n)$, limit D.N.E.

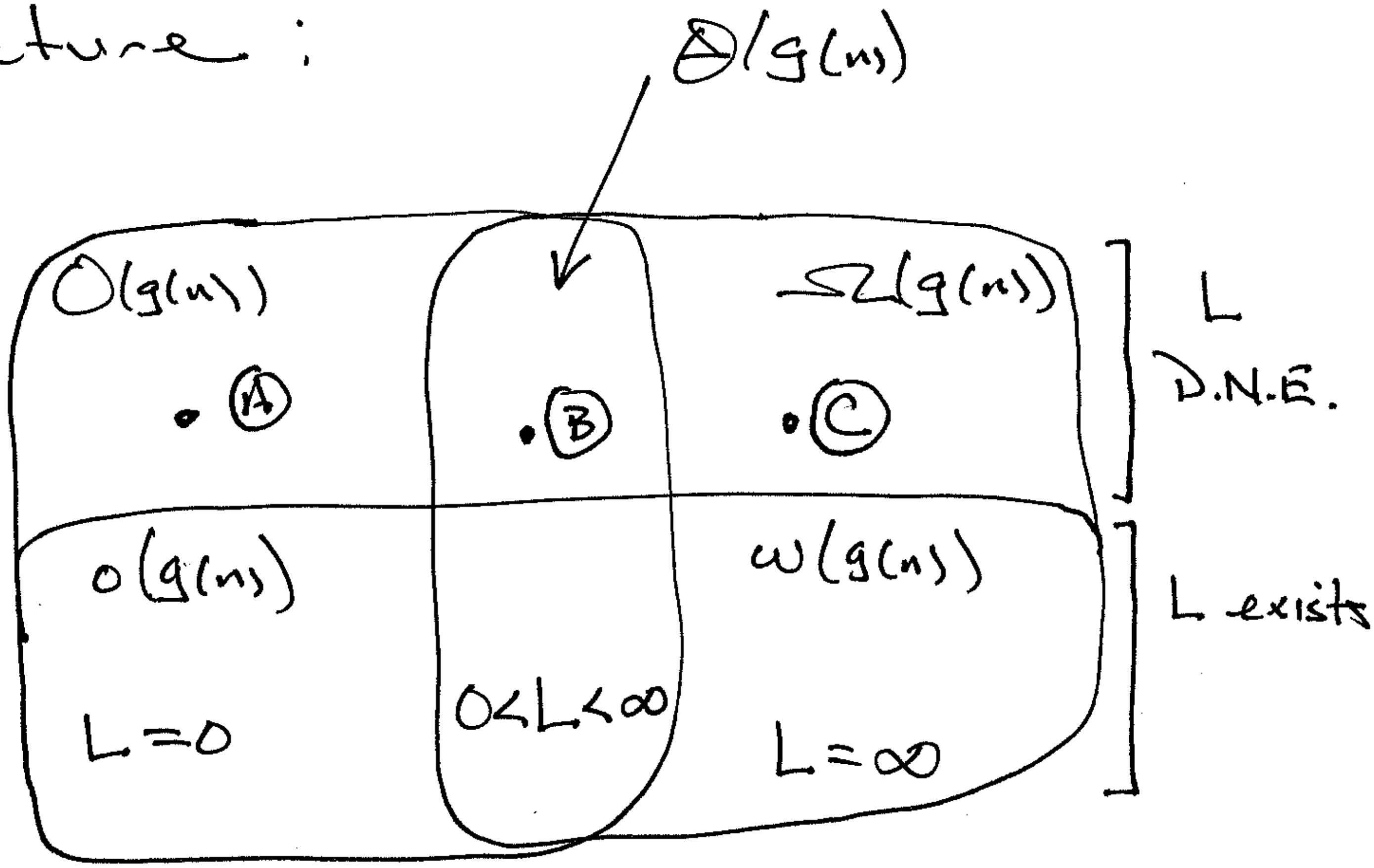
Exercise.

find example Ⓒ : functions

$f(n), g(n)$ s.t. $f(n) = \Omega(g(n))$,

$f(n) \neq O(g(n))$, But $\lim \frac{f(n)}{g(n)}$ D.N.E.

Picture :



<u>Recall</u> $a, b \in \mathbb{R},\ b > 0$:

$$(n+a)^b = \Theta(n^b)$$

<u>Proof</u>

$$\lim_{n\to\infty} \frac{(n+a)^b}{n^b} = \lim_{n\to\infty} \left(\frac{n+a}{n}\right)^b$$

$$= \left(\lim_{n\to\infty} \left(1+\frac{a}{n}\right)\right)^b$$

$$= 1^b = 1 \in (0,\infty)$$

$\therefore \quad (n+a)^b = \Theta(n^b)$ $\blacksquare$

Exercises

9c: $P(n)$ is polynomial of degree $k \geq 0$, then $P(n) = \Theta(n^k)$

Proof we have

$$P(n) = a_k \cdot n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$$

where $a_k > 0$. Therefore

$$\frac{P(n)}{n^k} = a_k + \underbrace{a_{k-1} \cdot \frac{1}{n}}_{\downarrow \, 0} + \cdots + \underbrace{a_0 \cdot \frac{1}{n^k}}_{\downarrow \, 0}$$

$$\therefore \lim_{n \to \infty} \left(\frac{P(n)}{n^k} \right) = a_k \in (0, \infty).$$

9d

Let $\alpha, \beta \in \mathbb{R}$

$$n^{\alpha} = \begin{cases} o(n^{\beta}) & \text{if } \alpha < \beta \\ \Theta(n^{\beta}) & \text{if } \alpha = \beta \\ \omega(n^{\beta}) & \text{if } \alpha > \beta \end{cases}$$

Proof.

$$\frac{n^{\alpha}}{n^{\beta}} = n^{\alpha - \beta} \xrightarrow[\text{as } n \to \infty]{} \begin{cases} 0 & \text{if } \alpha < \beta \\ 1 & \text{if } \alpha = \beta \\ \infty & \text{if } \alpha > \beta \end{cases}$$

9e let $a, b \in \mathbb{R}^+$.

$$a^n = \begin{cases} o(b^n) & \text{if } a < b \\ \Theta(b^n) & \text{if } a = b \\ \omega(b^n) & \text{if } a > b \end{cases}$$

Proof

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n \xrightarrow[n \to \infty]{\text{as}} \begin{cases} 0 & \text{if } a < b \\ 1 & \text{if } a = b \\ \infty & \text{if } a > b \end{cases}$$

$\blacksquare$

99.

$$f(n) + o(f(n)) = \Theta(f(n))$$

Proof

Let $h(n) = o(f(n))$, so

$$\lim_{n \to \infty} \left(\frac{h(n)}{f(n)} \right) = 0 .$$

Then

$$\frac{f(n)+h(n)}{f(n)} = 1 + \frac{h(n)}{f(n)} \longrightarrow 1 \in (0, \infty)$$

as $n \to \infty$

$\therefore \; f(n) + h(n) = \Theta(f(n))$.

$\blacksquare$

12 c

Compare 3^{2^n} to 2^{3^n}.

$$\lim_{n\to\infty}\left(\frac{3^{2^n}}{2^{3^n}}\right)=0$$

Trick

$$\ln\left(\frac{3^{2^n}}{2^{3^n}}\right)=\ln 3^{2^n}-\ln 2^{3^n}$$

$$=2^n\cdot\ln 3-3^n\cdot\ln 2$$

$$=3^n\left(\underbrace{\frac{2^n}{3^n}}_{\to\,0}\ln 3-\ln 2\right)\longrightarrow-\infty$$

$\therefore\quad 3^{2^n}=o\left(2^{3^n}\right)$

Some Common Functions

Let $a, b \in \mathbb{R}$, $a > 1$, $b > 1$.

$$a^{\log_a(x)} = x$$

$$\log_a(a^x) = x$$

$$\therefore \quad x = a^{\log_a(x)} = \left(b^{\log_b(a)}\right)^{\log_a(x)}$$

$$= b^{\log_b(a) \cdot \log_a(x)}$$

$$\therefore \quad \log_b(x) = \underbrace{\log_b(a)}_{\text{const.}} \cdot \log_a(x)$$

so

$$\log_b(n) = \text{const} \cdot \log_a(n)$$

$$\therefore \log_b(n) = \Theta(\log_a(n))$$

<u>Stirlings Formula</u>:

Let $n \in \mathbb{Z}^+$. Then

$$\boxed{n! = \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)}$$

<u>Corollary</u>

(a.) $n! = o(n^n)$

(b.) $n! = \omega(b^n)$ for any $b > 0$

(c.) $\log(n!) = \Theta(n \log n)$