3. The $n^{th}$ harmonic number is defined to be $H_n = \sum_{k=1}^{n} \left( \frac{1}{k} \right)$. Use induction to prove that

$$\sum_{k=1}^{n} H_k = (n + 1)H_n - n$$

for all $n \geq 1$. (Hint: Use the fact that $H_n = H_{n-1} + \frac{1}{n}$)

**Proof:**

I. If $n = 1$, then $H_1 = 1$ and $\sum_{k=1}^{1} H_k = 1 = 2 - 1 = (1 + 1) \cdot 1 - 1 = (1 + 1)H_1 - 1$, so the base case is satisfied.

II. Let $n > 1$ be chosen arbitrarily, and assume $\sum_{k=1}^{n-1} H_k = ((n - 1) + 1)H_{n-1} - (n - 1)$. We must show that $\sum_{k=1}^{n} H_k = (n + 1)H_n - n$. We have

$$\sum_{k=1}^{n} H_k = \sum_{k=1}^{n-1} H_k + H_n$$

$$= ((n - 1) + 1)H_{n-1} - (n - 1) + H_n \quad \text{by the induction hypothesis}$$

$$= nH_{n-1} - n + 1 + H_n$$

$$= nH_n - nH_n + nH_{n-1} - n + 1 + H_n$$

$$= (n + 1)H_n - n + 1 - n(H_n - H_{n-1})$$

$$= (n + 1)H_n - n + 1 - n \left( \frac{1}{n} \right) \quad \text{by the definition of } H_n$$

$$= (n + 1)H_n - n,$$

as required. It follows that $\sum_{k=1}^{n} H_k = (n + 1)H_n - n$ for all $n \geq 1$. ■

4. Define the sequence $T(n)$ for $n \geq 1$ by the recurrence $T(n) = (n - 1) + \frac{n - 1}{n^2} \cdot \sum_{k=1}^{n-1} T(k)$. Use induction to prove $T(n) \leq 2n$ for all $n \geq 1$.

**Proof:**

I. Observe $T(1) = (1 - 1) + \frac{1 - 1}{1^2} \cdot \text{(empty sum)} = 0 \leq 2 = 2 \cdot 1$, establishing the base case.

II. Let $n > 1$, and assume for all $k$ in the range $1 \leq k < n$ that $T(k) \leq 2k$. We must show that the inequality $T(n) \leq 2n$ is also true. We have

$$T(n) = (n - 1) + \frac{n - 1}{n^2} \cdot \sum_{k=1}^{n-1} T(k)$$
\[ \leq (n - 1) + \frac{n-1}{n^2} \cdot \sum_{k=1}^{n-1} 2k \quad \text{by the induction hypothesis} \]
\[ = (n - 1) + \frac{n-1}{n^2} \cdot 2 \cdot \frac{n(n-1)}{2} \]
\[ = (n - 1) + \frac{(n-1)^2}{n} \]
\[ = (n - 1) \left( 1 + \frac{n-1}{n} \right) \]
\[ = (n - 1) \left( 1 + 1 - \frac{1}{n} \right) \]
\[ = (n - 1) \left( 2 - \frac{1}{n} \right) \]
\[ = 2n - 2 - 1 + \frac{1}{n} \]
\[ = 2n - 3 + \frac{1}{n} \leq 2n \quad \text{(since } n > 1 \Rightarrow \frac{1}{n} \leq 1 \Rightarrow -3 + \frac{1}{n} \leq 0) \]

as required. It follows that \( T(n) \leq 2n \) for all \( n \geq 1 \). \[\blacksquare\]

5. Let \( T(n) \) satisfy the recurrence \( T(n) = aT(n/b) + f(n) \), where \( a \geq 1, b > 1 \) and \( f(n) \) is a polynomial satisfying \( \deg(f) > \log_b(a) \). Prove that case (3) of the Master Theorem applies, and in particular, prove that the regularity condition necessarily holds.

**Proof:**
Let \( d = \deg(f) \) and replace \( f(n) \) by the asymptotically equivalent function \( n^d \). We compare the polynomials \( n^d \) and \( n^{\log_b(a)} \). Let \( \epsilon = d - \log_b(a) \), which is positive since \( d > \log_b(a) \). Therefore \( d = \log_b(a) + \epsilon \), and \( n^d = \Omega(n^d) = \Omega(n^{\log_b(a)+\epsilon}) \), verifying the first hypothesis of case (3).

Observe \( d > \log_b(a) \Rightarrow b^d > a \Rightarrow a/b^d < 1 \). Pick any \( c \) in the range \( a/b^d \leq c < 1 \). Then for any \( n \geq 1 \), we have \( a(n/b)^d = (a/b^d)n^d \leq cn^d \), verifying the regularity condition. \[\blacksquare\]

6. Define \( T(n) \) by the recurrence
\[ T(n) = \begin{cases} 
0 & n = 1 \\
2T([n/2]) + n \lg(n) & n \geq 2 
\end{cases} \]
Here \( \lg \) means \( \log_2 \).

a. Show that the Master Theorem cannot be applied to this recurrence.

**Proof:**
We first simplify the recurrence to \( T(n) = 2T(n/2) + n \lg(n) \). Comparing \( n \lg(n) \) to \( n^{\log_2(2)} = n \), we see that \( \frac{n \lg(n)}{n} = \lg(n) \to \infty \) as \( n \to \infty \), so that \( n \lg(n) = \omega(n) \). Since \( n \lg(n) \) is the winner, the only possible case of the Master Theorem that could apply is case 3. But although \( n \lg(n) \) is the winner, it does not win by a polynomial factor. Indeed, let \( \epsilon > 0 \) be chosen arbitrarily. Then
\[
\frac{n \lg (n)}{n^{\log_2(2)+\epsilon}} = \frac{n \lg (n)}{n^{1+\epsilon}} = \frac{\lg(n)}{n^\epsilon} \rightarrow 0 \text{ as } n \rightarrow \infty,
\]

which yields \( n \lg(n) = o(n^{\log_2(2)+\epsilon}) \). Exercise 6 on page 5 of the handout on asymptotic growth rates implies \( o(n^{\log_2(2)+\epsilon}) \cap \Omega(n^{\log_2(2)+\epsilon}) = \emptyset \), and hence \( n \lg(n) \notin \Omega(n^{\log_2(2)+\epsilon}) \). Since \( \epsilon > 0 \) was arbitrary, this holds for all such \( \epsilon \). Therefore we are not in case 3, and the Master Theorem cannot be applied.

b. Use the Substitution method to prove that \( T(n) = O(n \left( \lg n \right)^2) \).

**Proof:**
We show by induction that \( T(n) \leq n (\lg n)^2 \) for all \( n \geq 1 \), from which \( T(n) = O(n (\lg n)^2) \) follows.

I. For \( n = 1 \) we have \( T(1) = 0 \leq 0 = 1 \cdot (\lg 1)^2 \), and the base case is satisfied.

II. Let \( n > 1 \) be arbitrary, and assume \( T(k) \leq k (\lg k)^2 \) for any \( k \) in the range \( 1 \leq k < n \). We must show that \( T(n) \leq n (\lg n)^2 \).

\[
T(n) = 2T([n/2]) + n \lg (n)
\]
\[
\leq 2 \cdot [n/2]([\lg(n/2)]^2 + n \lg(n)) \text{ By the induction hypothesis with } k = [n/2].
\]
\[
\leq 2 \cdot (n/2)(\lg(n/2))^2 + n \lg(n) \text{ Since } |x| \leq x \text{ for any } x \in \mathbb{R}
\]
\[
= n((\lg(n) - 1)^2 + n \lg(n)
\]
\[
= n((\lg(n))^2 - 2 \lg(n) + 1) + n \lg(n)
\]
\[
= n(\lg(n))^2 - 2n \lg(n) + n + n \lg(n)
\]
\[
= n(\lg(n))^2 - n \lg(n) + n
\]
\[
\leq n(\lg(n))^2
\]

The last inequality follows from

\[
n > 1 \Rightarrow n \geq 2 \Rightarrow \lg(n) \geq 1 \Rightarrow n \lg(n) \geq n \Rightarrow -n \lg(n) + n \leq 0.
\]

Therefore \( T(n) \leq n(\lg n)^2 \) for all \( n \geq 1 \) by the 2\(^{\text{nd}}\) PMI.