CSE 102
Midterm 1
Review Problems

1. Let a, b be real numbers, with $b > 0$. Prove that $(n+a)^b = \Theta(n^b)$.

2. Prove that $o(g(n)) \cap \Omega(g(n)) = \emptyset$ by verifying that no function can belong to both $o(g(n))$ and $\Omega(g(n))$.

3. The n^{th} harmonic number is defined to be $H_n = \sum_{k=1}^{n} \left(\frac{1}{k}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}$. Use induction to prove that

$$\sum_{k=1}^{n} H_k = (n+1)H_n - n$$

for all $n \geq 1$. (Hint: Use the fact that $H_n = H_{n-1} + \frac{1}{n}$.)

4. Define the sequence $T(n)$ for $n \geq 1$ by the recurrence $T(n) = (n-1) + \frac{n-1}{n^2} \cdot \sum_{k=1}^{n-1} T(k)$. Use induction to prove $T(n) \leq 2n$ for all $n \geq 1$.

5. Let $T(n)$ satisfy the recurrence $T(n) = aT(n/b) + f(n)$, where $a \geq 1$, $b > 1$ and $f(n)$ is a polynomial satisfying $\deg(f) > \log_b(a)$. Prove that case (3) of the Master Theorem applies, and in particular, prove that the regularity condition necessarily holds.

6. Define $T(n)$ by the recurrence

$$T(n) = \begin{cases} 0 & n = 1 \\ 2T(\lfloor n/2 \rfloor) + n \lg(n) & n \geq 2 \end{cases}$$

Here lg means $\log_2$.

   a. Show that the Master Theorem cannot be applied to this recurrence.
   b. Use the Substitution method to prove that $T(n) = O(n (\lg n)^2)$.