CSE 102
Spring 2024
Midterm Exam 1 **Solutions**

1. (20 Points) Prove that if $h_1(n) = \Theta\big(f(n)\big)$ and $h_2(n) = \Theta\big(g(n)\big)$, then $h_1(n)h_2(n) = \Theta\big(f(n)g(n)\big)$.

Proof:
We have:

$\exists$ positive a_1, b_1, n_1 such that $\forall n \geq n_1$: $0 \leq a_1 f(n) \leq h_1(n) \leq b_1 f(n)$
$\exists$ positive a_2, b_2, n_2 such that $\forall n \geq n_2$: $0 \leq a_2 g(n) \leq h_2(n) \leq b_2 g(n)$

Define $a = a_1 a_2$, $b = b_1 b_2$ and $n_0 = \max(n_1, n_2)$. Then a, b and n_0 are positive. If $n \geq n_0$, then both of the above inqualities are true. Upon multiplying these inequalities, we get

$\exists$ positive a, b, n_0 such that $\forall n \geq n_0$: $0 \leq af(n)g(n) \leq h_1(n)h_2(n) \leq bf(n)g(n)$

showing that $h_1(n)h_2(n) = \Theta\big(f(n)g(n)\big)$. ■

Note to graders:
It is wrong to conclude that since $h_1(n) = \Theta\big(f(n)\big)$, then the limit $\lim_{n\to\infty} h_1(n)/f(n)$ exists and is a positive real number, and similarly for $h_2(n)$ and $g(n)$. Anyone doing this should not receive full credit. ■

2. (20 Points) Use Stirling's formula to prove that $\frac{(3n)!}{(n!)^3} = \Theta\left(\frac{27^n}{n}\right)$.

Proof:

$$\frac{(3n)!}{(n!)^3} = \frac{\sqrt{2\pi \cdot 3n} \cdot \left(\frac{3n}{e}\right)^{3n} \cdot \left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\left(\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)\right)^3}$$

$$= \frac{\sqrt{3}}{2\pi} \cdot \frac{1}{n} \cdot \frac{3^{3n} \cdot n^{3n} \cdot e^{-3n}}{n^{3n} \cdot e^{-3n}} \cdot \frac{\left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\left(1 + \Theta\left(\frac{1}{n}\right)\right)^3}$$

$$= \frac{\sqrt{3}}{2\pi} \cdot \frac{27^n}{n} \cdot \frac{\left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\left(1 + \Theta\left(\frac{1}{n}\right)\right)^3}$$

Therefore

$$\frac{\frac{(3n)!}{(n!)^3}}{\frac{27^n}{n}} = \frac{\sqrt{3}}{2\pi} \cdot \frac{\left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\left(1 + \Theta\left(\frac{1}{n}\right)\right)^3} \;\rightarrow\; \frac{\sqrt{3}}{2\pi} \quad \text{as } n \rightarrow \infty$$

Since $0 < \sqrt{3}/2\pi < \infty$, it follows that $\frac{(3n)!}{(n!)^3} = \Theta\left(\frac{27^n}{n}\right)$. ■

3. (20 Points) The n^{th} harmonic number is defined to be he sum $H_n = \sum_{k=1}^{n}\left(\frac{1}{k}\right)$. Use induction to prove that for all $n \geq 1$:

$$\sum_{k=1}^{n} H_k = (n+1)H_n - n$$

(Hint: Use the fact that H_n satisfies the recurrence relation $H_n = H_{n-1} + \frac{1}{n}$.)

Proof: (We use weak induction form IIb)

I. If $n = 1$, then $H_1 = 1$ and $\sum_{k=1}^{1} H_k = 1 = 2 - 1 = (1+1) \cdot 1 - 1 = (1+1)H_1 - 1$, so the base case is satisfied.

II. Let $n > 1$ be chosen arbitrarily, and assume $\sum_{k=1}^{n-1} H_k = ((n-1)+1)H_{n-1} - (n-1)$. We must show that $\sum_{k=1}^{n} H_k = (n+1)H_n - n$. We have

$$\sum_{k=1}^{n} H_k = \sum_{k=1}^{n-1} H_k \; + \; H_n$$

$$= \big((n-1)+1\big)H_{n-1} - (n-1) + H_n \qquad \text{by the induction hypothesis}$$

$$= nH_{n-1} - n + 1 + H_n$$

$$= n\left(H_n - \frac{1}{n}\right) - n + 1 + H_n \qquad \text{since } H_{n-1} = H_n - \frac{1}{n} \text{ by the recurrence}$$

$$= nH_n - 1 - n + 1 + H_n$$

$$= (n+1)H_n - n,$$

as required. If follows that $\sum_{k=1}^{n} H_k = (n+1)H_n - n$ for all $n \geq 1$. ■

4. (20 Points) Define $T(n)$ by the recurrence

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 4T(\lfloor n/2 \rfloor) + 2n^2 & \text{if } n \geq 2 \end{cases}$$

a. (10 Points) Determine $c > 0$ such that $T(n) \leq cn^2 \lg(n)$ for all $n \geq 1$, hence $T(n) = O(n^2 \log(n))$.

Solution:
Let $c = 2$. We show by induction that $\forall n \geq 1$: $T(n) \leq 2n^2 \lg(n)$, from which $T(n) = O(n^2 \log(n))$ follows.

I. For $n = 1$ we have $T(1) = 0 \leq 0 = 2 \cdot 1^2 \lg(1)$, establishing the base case

II. Let $n > 1$ be chosen arbitrarily, and assume $T(k) \leq 2k^2 \lg(k)$ for k in the range $1 \leq k < n$. We must show that $T(n) \leq 2n^2 \lg(n)$. Then

$$\begin{aligned}
T(n) &= 4T\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) + 2n^2 && \text{by the definition of } T(n) \\
&\leq 4 \cdot 2 \left\lfloor \tfrac{n}{2} \right\rfloor^2 \lg \left\lfloor \tfrac{n}{2} \right\rfloor + 2n^2 && \text{by the induction hypothesis with } k = \lfloor n/2 \rfloor \\
&\leq 8 \left(\tfrac{n}{2}\right)^2 \lg\left(\tfrac{n}{2}\right) + 2n^2 && \text{since } \lfloor x \rfloor \leq x \text{ for any } x \in \mathbb{R} \\
&= 2n^2(\lg(n) - 1) + 2n^2 \\
&= 2n^2 \lg(n) - 2n^2 + 2n^2 \\
&= 2n^2 \lg(n)
\end{aligned}$$

The result follows for all $n \geq 1$ by the 2nd PMI. ■

b. (10 Points) Use the Master Theorem to find a tight asymptotic bound for $T(n)$.

Solution:
Simplifying as appropriate for the Master Theorem gives $T(n) = 4T(n/2) + n^2$. We compare n^2 to $n^{\log_2(4)} = n^2$. Case 2 yields $T(n) = \Theta(n^2 \log(n))$. ■

5. (20 Points) Use the Master Theorem to find a tight asymptotic bound for the solution to the following recurrence relation.

$$T(n) = 25\,T(n/3) + n^3$$

Solution:
We compare n^3 to $n^{\log_3(25)}$. Since $25 < 27 = 3^3$ we have $\log_3(25) < 3$. Let $\epsilon = 3 - \log_3(25)$. Then $\epsilon > 0$ and $3 = \log_3(25) + \epsilon$. Therefore $n^3 = \Omega(n^3) = \Omega\left(n^{\log_3(25)+\epsilon}\right)$, putting us in case 3. To establish the regularity condition, choose any c in the range $\frac{25}{27} \le c < 1$. Then for any $n \ge 1$ we have

$$25\left(\frac{n}{3}\right)^3 = \frac{25}{27} \cdot n^3 \le cn^3.$$

It now follows from case 3 of the Master Theorem that $T(n) = \Theta(n^3)$. ■