CSE 102
Midterm 1 Review
Solutions to Selected Problems

3. The n^{th} harmonic number is defined to be $H_n = \sum_{k=1}^{n}\left(\frac{1}{k}\right)$. Use induction to prove that

$$\sum_{k=1}^{n} H_k = (n+1)H_n - n$$

for all $n \geq 1$. (Hint: Use the fact that $H_n = H_{n-1} + \frac{1}{n}$.)

Proof:

I. If $n = 1$, then $H_1 = 1$ and $\sum_{k=1}^{1} H_k = 1 = 2 - 1 = (1+1)\cdot 1 - 1 = (1+1)H_1 - 1$, so the base case is satisfied.

II. Let $n > 1$ be chosen arbitrarily, and assume $\sum_{k=1}^{n-1} H_k = ((n-1)+1)H_{n-1} - (n-1)$. We must show that $\sum_{k=1}^{n} H_k = (n+1)H_n - n$. We have

$$\sum_{k=1}^{n} H_k = \sum_{k=1}^{n-1} H_k \; + \; H_n$$

$$= \big((n-1)+1\big)H_{n-1} - (n-1) + H_n \qquad \text{by the induction hypothesis}$$

$$= nH_{n-1} - n + 1 + H_n$$

$$= nH_n - nH_n + nH_{n-1} - n + 1 + H_n$$

$$= (n+1)H_n - n + 1 - n(H_n - H_{n-1})$$

$$= (n+1)H_n - n + 1 - n\left(\frac{1}{n}\right) \qquad \text{by the definition of } H_n$$

$$= (n+1)H_n - n,$$

as required. If follows that $\sum_{k=1}^{n} H_k = (n+1)H_n - n$ for all $n \geq 1$. ∎

4. Define the sequence $T(n)$ for $n \geq 1$ by the recurrence $T(n) = (n-1) + \frac{n-1}{n^2} \cdot \sum_{k=1}^{n-1} T(k)$. Use induction to prove $T(n) \leq 2n$ for all $n \geq 1$.

Proof:

I. Observe $T(1) = (1-1) + \frac{1-1}{1^2} \cdot (\text{empty sum}) = 0 \leq 2 = 2 \cdot 1$, establishing the base case.

II. Let $n > 1$, and assume for all k in the range $1 \leq k < n$ that $T(k) \leq 2k$. We must show that the inequality $T(n) \leq 2n$ is also true. We have

$$T(n) = (n-1) + \frac{n-1}{n^2} \cdot \sum_{k=1}^{n-1} T(k)$$

$$
\begin{aligned}
&\leq (n-1) + \frac{n-1}{n^2} \cdot \textstyle\sum_{k=1}^{n-1} 2k && \text{by the induction hypothesis} \\
&= (n-1) + \frac{n-1}{n^2} \cdot 2 \cdot \frac{n(n-1)}{2} \\
&= (n-1) + \frac{(n-1)^2}{n} \\
&= (n-1)\left(1 + \frac{n-1}{n}\right) \\
&= (n-1)\left(1 + 1 - \frac{1}{n}\right) \\
&= (n-1)\left(2 - \frac{1}{n}\right) \\
&= 2n - 2 - 1 + \frac{1}{n} \\
&= 2n - 3 + \frac{1}{n} \leq 2n && \left(\text{since } n > 1 \Rightarrow \frac{1}{n} \leq 1 \Rightarrow -3 + \frac{1}{n} \leq 0\right)
\end{aligned}
$$

as required. It follows that $T(n) \leq 2n$ for all $n \geq 1$. ■

5. Let $T(n)$ satisfy the recurrence $T(n) = aT(n/b) + f(n)$, where $a \geq 1$, $b > 1$ and $f(n)$ is a polynomial satisfying $\deg(f) > \log_b(a)$. Prove that case (3) of the Master Theorem applies, and in particular, prove that the regularity condition necessarily holds.

   Proof:
   Let $d = \deg(f)$ and replace $f(n)$ by the asymptotically equivalent function n^d. We compare the polynomials n^d and $n^{\log_b(a)}$. Let $\epsilon = d - \log_b(a)$, which is positive since $d > \log_b(a)$. Therfore $d = \log_b(a) + \epsilon$, and $n^d = \Omega(n^d) = \Omega(n^{\log_b(a)+\epsilon})$, verifying the first hypothesis of case (3).

   Observe $d > \log_b(a) \Rightarrow b^d > a \Rightarrow a/b^d < 1$. Pick any c in the range $a/b^d \leq c < 1$. Then for any $n \geq 1$, we have $a(n/b)^d = (a/b^d)n^d \leq cn^d$, verifying the regularity condition. ■

6. Define $T(n)$ by the recurrence

$$
T(n) = \begin{cases} 0 & n = 1 \\ 2T(\lfloor n/2 \rfloor) + n \lg(n) & n \geq 2 \end{cases}
$$

   Here lg means $\log_2$.

   a. Show that the Master Theorem cannot be applied to this recurrence.

   Proof:
   We first simplify the recurrence to $T(n) = 2T(n/2) + n\lg(n)$. Comparing $n \lg(n)$ to $n^{\log_2(2)} = n$, we see that $\frac{n\lg(n)}{n} = \lg(n) \longrightarrow \infty$ as $n \to \infty$, so that $n\lg(n) = \omega(n)$. Since $n\lg(n)$ is the winner, the only possible case of the Master Theorem that could apply is case 3. But although $n\lg(n)$ is the winner, it does not win by a polynomial factor. Indeed, let $\epsilon > 0$ be chosen arbitrarily. Then

$$\frac{n \lg(n)}{n^{\log_2(2)+\epsilon}} = \frac{n \lg(n)}{n^{1+\epsilon}} = \frac{\lg(n)}{n^{\epsilon}} \longrightarrow 0 \text{ as } n \to \infty,$$

which yeilds $n \lg(n) = o\left(n^{\log_2(2)+\epsilon}\right)$. Exercise 6 on page 5 of the handout on asymptotic growth rates implies $o\left(n^{\log_2(2)+\epsilon}\right) \cap \Omega\left(n^{\log_2(2)+\epsilon}\right) = \emptyset$, and hence $n \lg(n) \notin \Omega\left(n^{\log_2(2)+\epsilon}\right)$. Since $\epsilon > 0$ was arbitrary, this holds for *all* such ϵ. Therefore we are not in case 3, and the Master Theorem cannot be applied. ■

b. Use the Substitution method to prove that $T(n) = O(n\,(\lg n)^2)$.

Proof:
We show by induction that $T(n) \le n(\lg n)^2$ for all $n \ge 1$, from which $T(n) = O(n\,(\lg n)^2)$ follows.

I. For $n = 1$ we have $T(1) = 0 \le 0 = 1 \cdot (\lg 1)^2$, and the base case is satisfied.

II. Let $n > 1$ be arbitrary, and assume $T(k) \le k(\lg k)^2$ for any k in the range $1 \le k < n$. We must show that $T(n) \le n(\lg n)^2$.

$$T(n) = 2T(\lfloor n/2 \rfloor) + n \lg(n)$$

$$\le 2 \cdot \lfloor n/2 \rfloor (\lg \lfloor n/2 \rfloor)^2 + n \lg(n) \qquad \text{By the induction hypothesis with } k = \lfloor n/2 \rfloor.$$

$$\le 2 \cdot (n/2)(\lg(n/2))^2 + n \lg(n) \qquad \text{Since } \lfloor x \rfloor \le x \text{ for any } x \in \mathbb{R}$$

$$= n(\lg(n) - 1)^2 + n \lg(n)$$

$$= n((\lg(n))^2 - 2\lg(n) + 1) + n \lg(n)$$

$$= n(\lg n)^2 - 2n\lg(n) + n + n \lg(n)$$

$$= n(\lg n)^2 - n\lg(n) + n$$

$$\le n(\lg n)^2$$

The last inequality follows from

$$n > 1 \;\Rightarrow\; n \ge 2 \;\Rightarrow\; \lg(n) \ge 1 \;\Rightarrow\; n\lg(n) \ge n \;\Rightarrow\; -n\lg(n) + n \le 0.$$

Therefore $T(n) \le n(\lg n)^2$ for all $n \ge 1$ by the 2nd PMI. ■