

CSE 102

Homework Assignment 1

Solutions

1. (Problem 3.1-1) Let $f(n)$ and $g(n)$ asymptotically positive functions. Prove that $f(n) + g(n) = \Theta(\max(f(n), g(n)))$.

Proof:

Since $f(n)$ and $g(n)$ are asymptotically positive, we know that there exists a positive constant n_0 such that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$. For any such n we have

$$\begin{aligned} 0 &\leq \max(f(n), g(n)) \\ &\leq \min(f(n), g(n)) + \max(f(n), g(n)) \\ &\leq 2 \cdot \max(f(n), g(n)). \end{aligned}$$

But $f(n) + g(n) = \min(f(n), g(n)) + \max(f(n), g(n))$, and therefore

$$0 \leq 1 \cdot \max(f(n), g(n)) \leq f(n) + g(n) \leq 2 \cdot \max(f(n), g(n))$$

for all $n \geq n_0$, showing that $f(n) + g(n) = \Theta(\max(f(n), g(n)))$. ■

2. Prove or disprove: If $f(n) = \Theta(g(n))$, then $f(n)^2 = \Theta(g(n)^2)$.

Proof:

Since $f(n) = \Theta(g(n))$, there are positive constants c_1, c_2 and n_0 such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

for all $n \geq n_0$. Squaring this inequality gives $0 \leq c_1^2 g(n)^2 \leq f(n)^2 \leq c_2^2 g(n)^2$ for all $n \geq n_0$. Thus there exist positive constants d_1, d_2 and n_1 (namely $d_1 = c_1^2, d_2 = c_2^2$ and $n_1 = n_0$) such that

$$0 \leq d_1 g(n)^2 \leq f(n)^2 \leq d_2 g(n)^2$$

for all $n \geq n_1$. This proves $f(n)^2 = \Theta(g(n)^2)$. ■

3. Prove or disprove: If $f(n) = \Theta(g(n))$, then $2^{f(n)} = \Theta(2^{g(n)})$.

Counter Example:

Let $f(n) = 2n$ and $g(n) = n$. Then $f(n) = \Theta(g(n))$, but $2^{2n} = 4^n = \omega(2^n)$, and therefore $2^{f(n)} = \omega(2^{g(n)})$, whence $2^{f(n)} \neq \Theta(2^{g(n)})$. ■

4. Let $f(n)$ and $g(n)$ be asymptotically positive functions, and assume that $\lim_{n \rightarrow \infty} g(n) = \infty$. Prove that if $f(n) = \Theta(g(n))$, then $\ln(f(n)) = \Theta(\ln(g(n)))$.

Proof:

Assume $f(n) = \Theta(g(n))$. Then there exist positive constants c_1, c_2 and n_0 such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

for all $n \geq n_0$. Since $\lim_{n \rightarrow \infty} g(n) = \infty$, the constant n_0 can be chosen large enough so that

$$1 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

for all $n \geq n_0$. Take $\ln()$ of all terms in the preceding inequality to get

$$0 \leq \ln(c_1) + \ln(g(n)) \leq \ln(f(n)) \leq \ln(c_2) + \ln(g(n))$$

for all $n \geq n_0$. Since $\lim_{n \rightarrow \infty} g(n) = \infty$, the term $\ln(g(n))$ dominates the constants $\ln(c_1)$ and $\ln(c_2)$, whence $\ln(c_1) + \ln(g(n)) = \Omega(\ln(g(n)))$ and $\ln(c_2) + \ln(g(n)) = O(\ln(g(n)))$. The desired result $\ln(f(n)) = \Theta(\ln(g(n)))$ now follows by Exercise 4 on page 4 of the handout on asymptotic growth rates. ■

5. (Problem 3.2-8) Show that if $f(n) \ln f(n) = \Theta(n)$, then $f(n) = \Theta(n/\ln n)$. Hint: use the result of the preceding problem.

Proof:

Assume $f(n) \ln f(n) = \Theta(n)$. Since $\lim_{n \rightarrow \infty} n = \infty$, we can, by the result of problem (4), take $\ln()$ of both sides of this equation to get

$$\ln(f(n)) + \ln(\ln(f(n))) = \Theta(\ln(n)).$$

Since the term $\ln(\ln(f(n))) = o(\ln(f(n)))$, we can, by Exercise 10g on page 7 of the handout on asymptotic growth rates, drop the lower order term and write this as $\ln(f(n)) = \Theta(\ln(n))$. Substituting this into our assumption gives $f(n)\Theta(\ln n) = \Theta(n)$, and therefore

$$f(n) = \Theta(n) \cdot \frac{1}{\Theta(\ln n)} = \Theta(n) \cdot \Theta\left(\frac{1}{\ln n}\right) = \Theta\left(\frac{n}{\ln n}\right)$$

where we have used Exercise 12bc on page 8 of the handout on asymptotic growth rates. ■

6. Consider the statement: $f(cn) = \Theta(f(n))$.

a. Determine a function $f(n)$ and a constant $c > 0$ for which the statement is false.

Example:

Let $f(n) = 2^n$ and $c = 2$. Then the statement says $2^{2n} = \Theta(2^n)$, i.e. $4^n = \Theta(2^n)$. This is false since $4^n = \omega(2^n)$ by Exercise 10e on page 7 of the handout on Asymptotic Growth Rates, and since $\omega(2^n) \cap \Theta(2^n) = \emptyset$ by Exercise 7 on page 5 of the same handout. ■

b. Determine a function $f(n)$ for which the statement is true for all $c > 0$.

Example:

Let $f(n) = n$. Then the statement says $cn = \Theta(n)$. This is true for any $c > 0$ by Exercise 2 on page 3 of the handout on Asymptotic Growth Rates. ■

7. Determine the asymptotic order of the expression $\sum_{i=1}^n a^i$ where $a > 0$ is a constant, i.e. find a simple function $g(n)$ such that the expression is in the class $\Theta(g(n))$. (Hint: consider the cases $a = 1$, $a > 1$, and $0 < a < 1$ separately.)

Solution: In the case $a = 1$ we have $\sum_{i=1}^n a^i = n = \Theta(n)$. Now assume $a \neq 1$ and use the formula for the sum of a geometric series:

$$\sum_{i=1}^n a^i = a \left(\sum_{i=0}^{n-1} a^i \right) = a \left(\frac{a^n - 1}{a - 1} \right) = \left(\frac{a}{a - 1} \right) \cdot a^n - \left(\frac{a}{a - 1} \right)$$

If $a > 1$ then $a^n \rightarrow \infty$ as $n \rightarrow \infty$, so the first term dominates the second, and $\sum_{i=1}^n a^i = \Theta(a^n)$. If $0 < a < 1$ then $a^n \rightarrow 0$, so the constant term (which is positive in this case) dominates, and therefore $\sum_{i=1}^n a^i = \Theta(1)$. Thus we have

$$\sum_{i=1}^n a^i = \begin{cases} \Theta(1) & \text{if } 0 < a < 1 \\ \Theta(n) & \text{if } a = 1 \\ \Theta(a^n) & \text{if } a > 1 \end{cases}$$

■

8. Use induction to prove that $\sum_{k=1}^n k^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$ for all $n \geq 1$.

Proof:

I. $\sum_{k=1}^1 k^4 = 1 = \frac{2 \cdot 15}{30} = \frac{1 \cdot (1+1) \cdot (6 \cdot 1^3 + 9 \cdot 1^2 + 1 - 1)}{30}$, and the base case is established.

II. Let $n \geq 1$. Assume for this n that

$$\sum_{k=1}^n k^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$$

We must show that

$$\sum_{k=1}^{n+1} k^4 = \frac{(n+1)(n+2)(6(n+1)^3+9(n+1)^2+(n+1)-1)}{30}$$

We have

$$\begin{aligned} \sum_{k=1}^{n+1} k^4 &= \left(\sum_{k=1}^n k^4 \right) + (n+1)^4 \\ &= \frac{n(n+1)(6n^3+9n^2+n-1)}{30} + (n+1)^4 \quad \text{by the induction hypothesis} \\ &= \frac{(n+1)[n(6n^3+9n^2+n-1) + 30(n+1)^3]}{30} \\ &= \frac{(n+1)[6n^4+39n^3+91n^2+89n+30]}{30} \\ &= \frac{(n+1)(n+2)[6n^3+27n^2+37n+15]}{30} \\ &= \frac{(n+1)((n+1)+1)[6(n+1)^3+9(n+1)^2+(n+1)-1]}{30} \end{aligned}$$

Thus the formula holds for $n+1$ as well. The formula is valid for all n by induction. ■