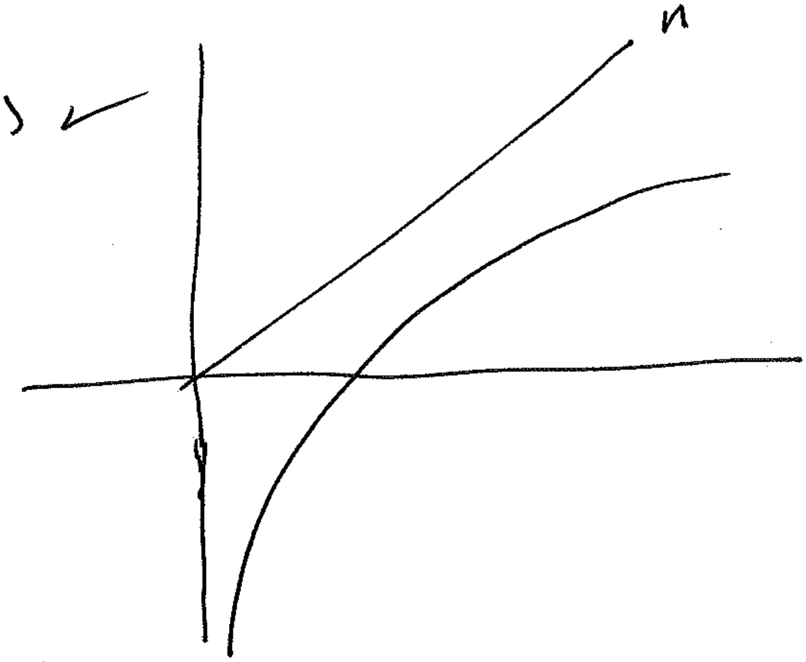
CSE 102 9-29-22

Ex. $\ln(n) = o(n)$ ✓



$$\lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right) = \lim_{n\to\infty}\left(\frac{1/n}{1}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

Exercise $\ln(n) = o(n^k)$ for any $k>0$

Exercise $(\ln(n))^m = o(n^k)$ for any $k>0$, $m\geq 0$

Exercise $(\log_b(n))^m = o(n^k)$ for any $\begin{cases} k>0 \\ m\geq 0 \\ b>1 \end{cases}$

Ex. $n^k = o(e^n)$ ✓ for any $k \in \mathbb{R}$

$$\lim_{n\to\infty}\left(\frac{n^k}{e^n}\right) = \lim_{n\to\infty}\left(\frac{kn^{k-1}}{e^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{k(k-1)n^{k-2}}{e^n}\right)$$

$$\vdots$$

$$= 0$$

after $\lceil k \rceil$ apps. of l'Hop.

Exercise $n^k = o(b^n)$ for any $k \in \mathbb{R}$, $b > 1$

Recall: $o(g(n)) \subseteq O(g(n))$

$\overleftarrow{O}, \overleftarrow{\Omega}, \overleftarrow{\Theta}, \overleftarrow{o}, \underline{\underline{\omega}}$

<u>Exercise</u>

show that $o(g(n)) \cap \Omega(g(n)) = \phi$

<u>Defn</u>

$$\omega(g(n)) = \{f(n) \mid \forall c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c \cdot g(n) < f(n)\}$$

<u>Recall</u>:

$$\Omega(g(n)) = \{f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c \cdot g(n) \leq f(n)\}$$

$$\therefore \quad \omega(g(n)) \subseteq \Omega(g(n)).$$

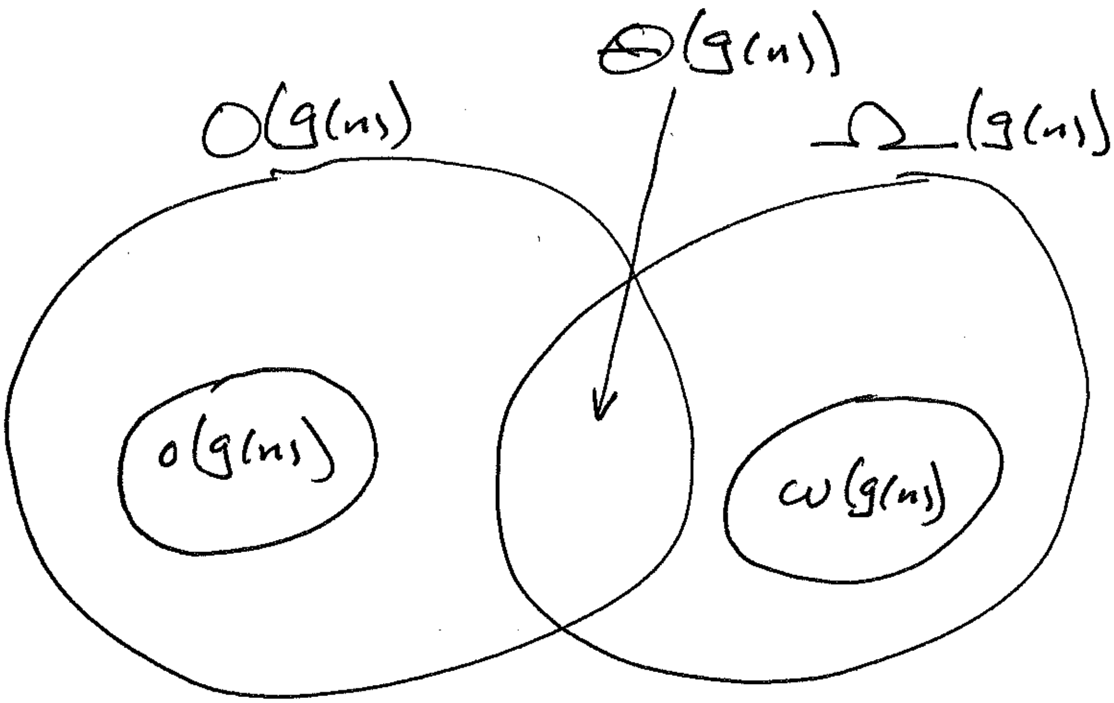
Exercise

Prove that

- $f(n) = \omega(g(n))$ iff $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = \infty$

- $f(n) = \omega(g(n))$ iff $g(n) = o(f(n))$

- $\omega(g(n)) \cap O(g(n)) = \phi$



Theorem

If $\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = L$, where $0 \le L < \infty$,

then $f(n) = O(g(n))$.

Rmk: The converse is false.

Proof

limit stmt. says

$$\forall \varepsilon > 0, \exists n_0 > 0, \forall n \ge n_0 : \left| \frac{f(n)}{g(n)} - L \right| < \varepsilon .$$

Let $\varepsilon = 1$. Then there exist $n_0 > 0$ s.t. for all $n \ge n_0$

$$\left| \frac{f(n)}{g(n)} - L \right| < 1$$

$$\therefore \quad -1 < \underbrace{\frac{f(n)}{g(n)} - L}_{} < 1$$

$$\therefore \quad \frac{f(n)}{g(n)} < 1 + L$$

$$\therefore \quad 0 \leq f(n) \leq (1+L)\, g(n)$$

Let $c = 1+L$. so there exist positive c, n_0 s.t. $\forall n \geq n_0$

$$0 \leq f(n) \leq c\, g(n)$$

holds. Hence $f(n) = O(g(n))$. ∎

<u>Theorem</u>

If $\lim\limits_{n\to\infty} \left(\dfrac{f(n)}{g(n)}\right) = L$, where $0 < L \leq \infty$,

then $f(n) = \Omega(g(n))$.

<u>Rmk</u> converse is false.

<u>Proof</u>

the limit implies $\lim\limits_{n\to\infty} \left(\dfrac{g(n)}{f(n)}\right) = L'$

where $L' = \dfrac{1}{L}$. Thus $0 \leq L' < \infty$.

$\therefore$ $g(n) = O(f(n))$. $\therefore$ $f(n) = \Omega(g(n))$.

$\blacksquare$

Theorem

If $\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = L$ where $0 < L < \infty$,

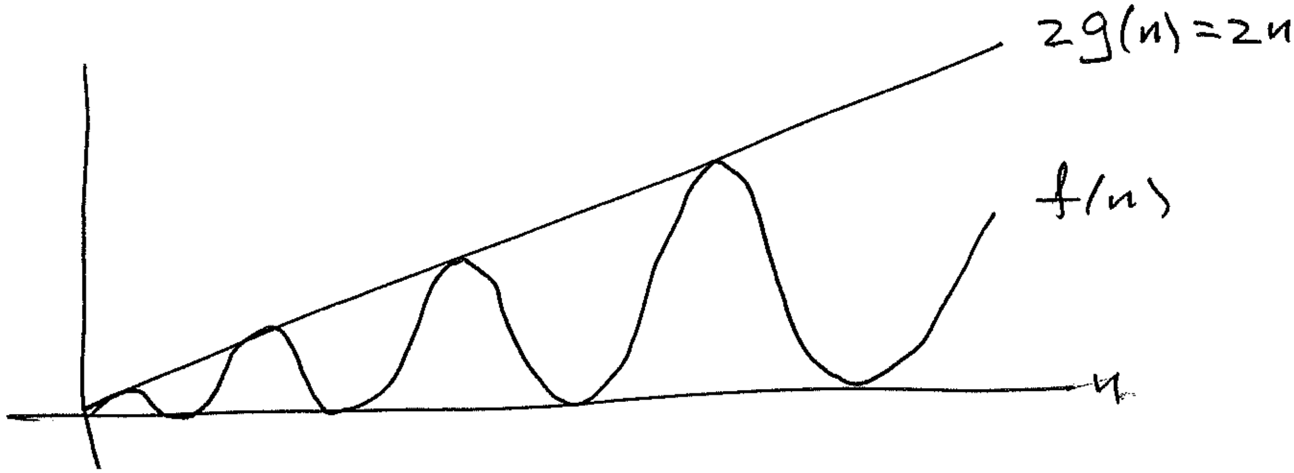
then $f(n) = \Theta(g(n))$

Proof: exercise.

Rmk: converse is false.

Example A

let $g(n) = n$, $f(n) = (1+\sin(n))n$.

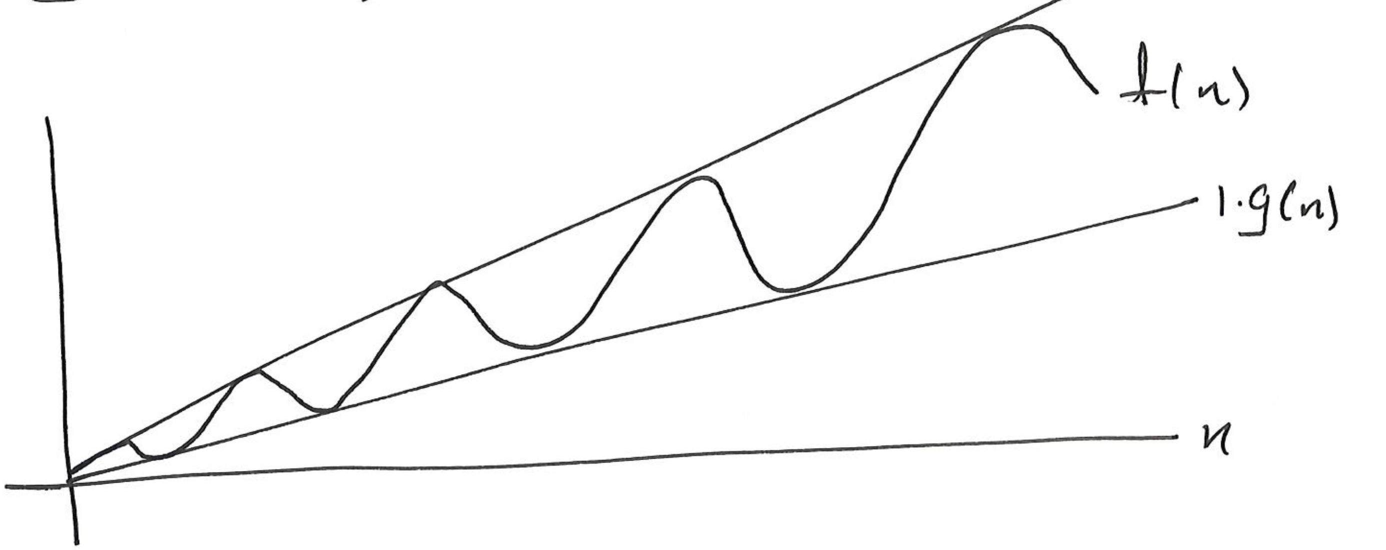


note: $\frac{f(n)}{g(n)} = 1+\sin(n) \longrightarrow$ no limit
D.N.E

obviously $f(n)=O(g(n))$. also notice $f(n)\neq\Omega(g(n))$.

Example B

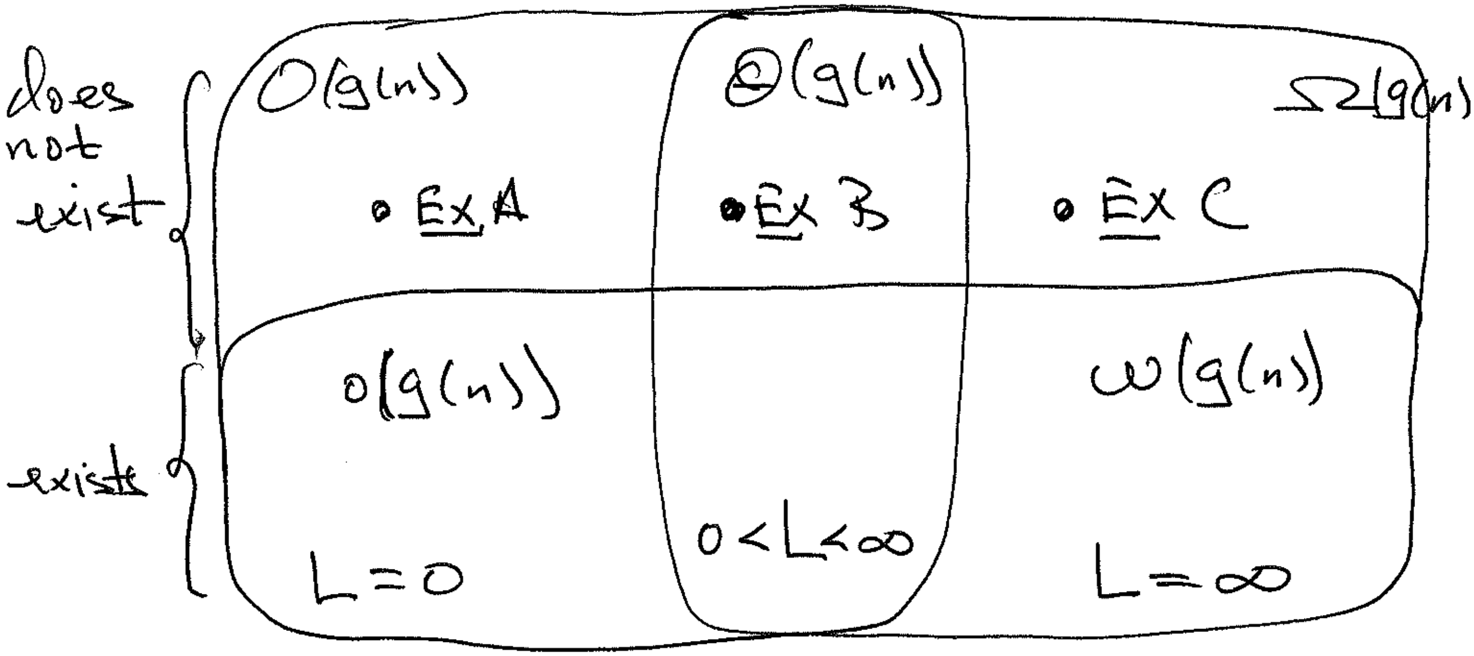
let $g(n)=n$, $f(n)=(2+\sin(n))\,n$



$\therefore\ f(n)=\Theta(g(n))$

But $\dfrac{f(n)}{g(n)} = 2+\sin(n) \longrightarrow$ limit D.N.E.

Picture



let $L = \lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right)$, if it exists.

<u>Exercise</u> find example C, i.e.

$f(n), g(n)$ s.t. $f(n) = \Omega(g(n))$,

$f(n) \neq O(g(n))$ but

$\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right)$ D.N.E.

Ex.

If $P(n)$ is a polynomial of degree $k \geq 0$, then $P(n) = \Theta(n^k)$

Proof.

Let $P(n) = a_k n^k + a_{k-1} \cdot n^{k-1} + \cdots + a_1 \cdot n + a_0$

where $a_k > 0$, and $a_{k-1}, \ldots, a_0 \in \mathbb{R}$.

Then

$$\frac{P(n)}{n^k} = a_k + a_{k-1} \cdot \underbrace{\frac{1}{n}}_{\downarrow \, 0} + \cdots + a_1 \cdot \underbrace{\frac{1}{n^{k-1}}}_{\downarrow \, 0} + a_0 \cdot \underbrace{\frac{1}{n^k}}_{\downarrow \, 0}$$

$\therefore \frac{P(n)}{n^k} \longrightarrow a_k$ as $n \to \infty$, and

$0 < a_k < \infty$. $\therefore P(n) = \Theta(n^k)$ ∎

Ex. let $\alpha, \beta \in \mathbb{R}$. then

$$n^\alpha = \begin{cases} o(n^\beta) & \text{if } \alpha < \beta \\ \Theta(n^\beta) & \text{if } \alpha = \beta \\ \omega(n^\beta) & \text{if } \alpha > \beta \end{cases}$$

Proof.

$$\frac{n^\alpha}{n^\beta} = n^{\alpha-\beta} \longrightarrow \begin{cases} 0 & \text{if } \alpha < \beta \\ 1 & \text{if } \alpha = \beta \\ \infty & \text{if } \alpha > \beta \end{cases}$$

∎

Ex. for any $a > 1$, $b > 1$:

$$a^n = \begin{cases} o(b^n) & \text{if } a < b \\ \Theta(b^n) & \text{if } a = b \\ \omega(b^n) & \text{if } a > b \end{cases}$$

Proof:

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n \longrightarrow \begin{cases} 0 & \text{if } a<b \\ 1 & \text{if } a=b \\ \infty & \text{if } a>b \end{cases}$$

■

Ex. $f(n) + o(f(n)) = \Theta(f(n))$.

Proof

let $h(n) = o(f(n))$. Then $\lim_{n\to\infty}\left(\frac{h(n)}{f(n)}\right) = 0$.

so

$$\frac{f(n)+h(n)}{f(n)} = 1 + \underbrace{\frac{h(n)}{f(n)}}_{\downarrow\, 0} \longrightarrow 1 \in (0,\infty)$$

$\therefore$ $f(n)+h(n) = \Theta(f(n))$.

■

analogy

$f(n) = O(g(n))$	$\approx$	$x \leq y$
Ω	...	$x \geq y$
Θ	...	$=$
o	...	$<$
ω	...	$>$