we defined

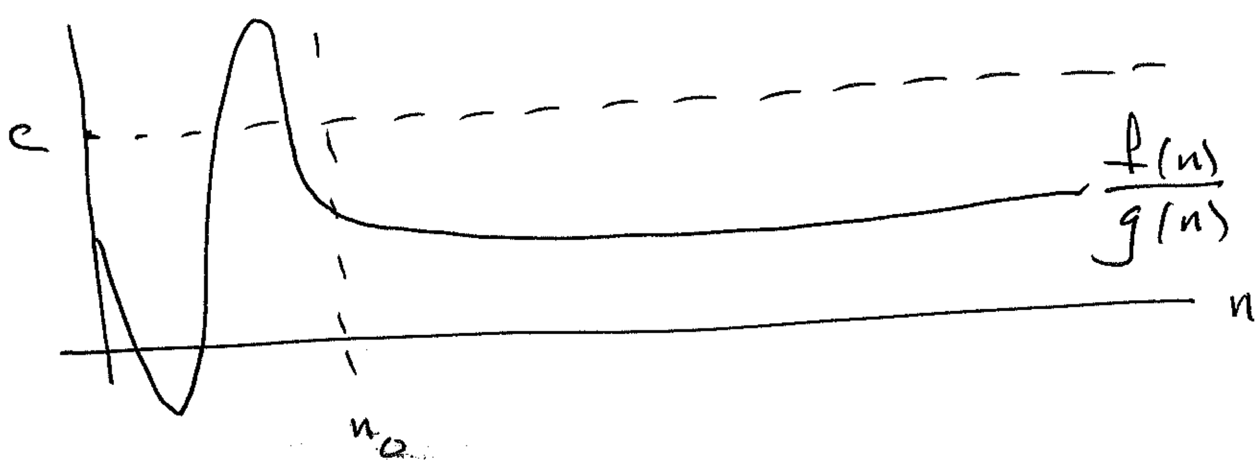
- $f(n) = O(g(n))$ iff

  $\exists$ pos. c, n_0 s.t. $\forall n \geq n_0$

  $$0 \leq f(n) \leq c \cdot g(n)$$

  note: if $g(n)$ is asymp. pos.,
  then this is equiv. to

  $$0 \leq \frac{f(n)}{g(n)} \leq c$$



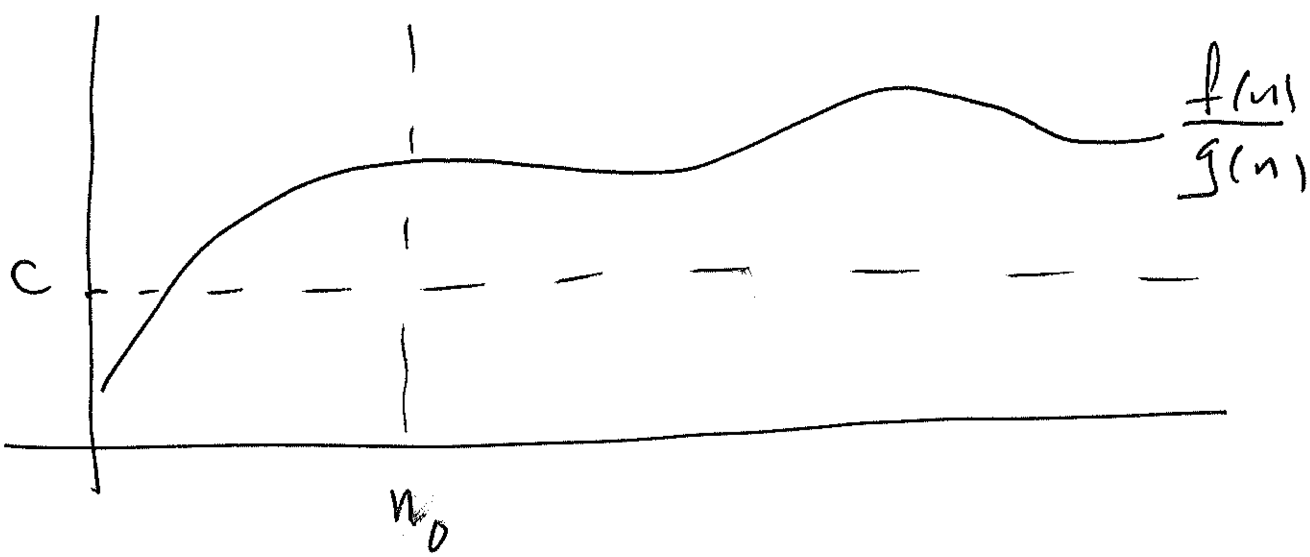
- $f(n) = \Omega(g(n))$ iff

$\exists$ pos. c, n_0 s.t. $\forall n \geq n_0$

$$0 \leq c \cdot g(n) \leq f(n)$$

note: if $g(n)$ is asym. pos. then this is equiv. to

$$0 \leq c \leq \frac{f(n)}{g(n)}$$



<u>note</u>: we make a blanket assumption that all fcns. are asym. non-negative.

<u>Defn</u>

$$\Theta(g(n)) = \{ f(n) \mid \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0 \; \forall n \geq n_0 : 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \}$$

equivalently: $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$

say $g(n)$ is a <u>tight</u> asymptotic bound on $f(n)$.

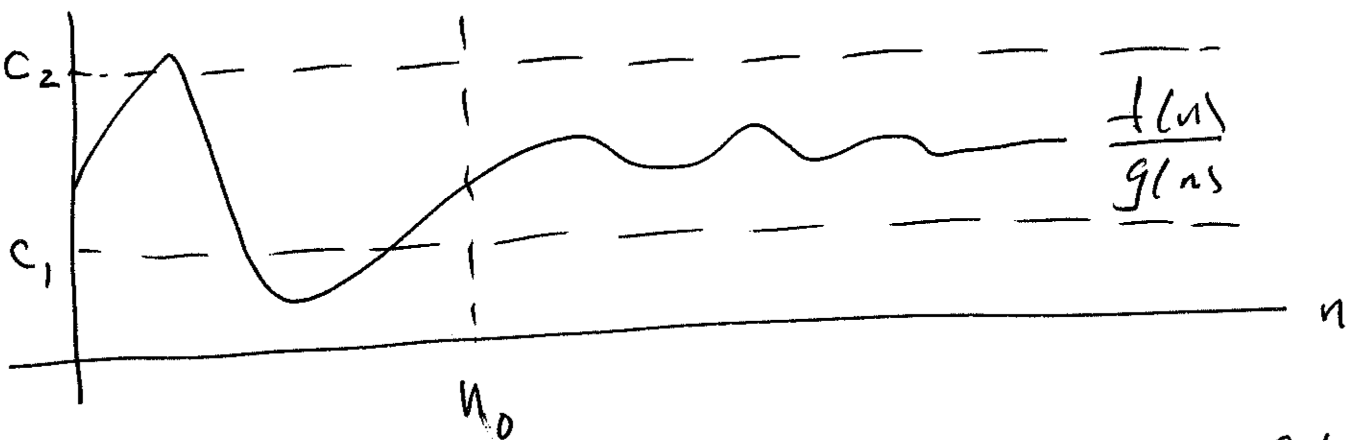
<u>notation</u>: $f(n) = \Theta(g(n))$.

- equivalent definition:

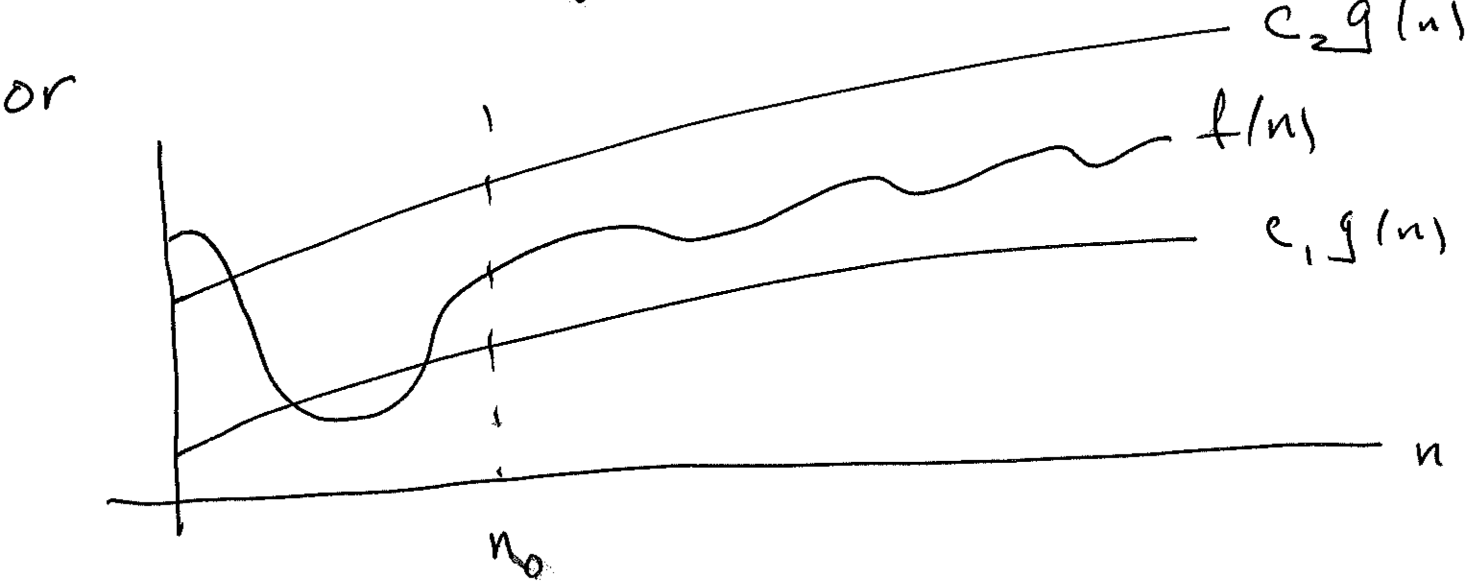
$\exists$ pos. c_1, c_2, n_0 s.t. $\forall n \geq n_0$

$$0 \leq c_1 \leq \frac{f(n)}{g(n)} \leq c_2$$

Provided $g(n)$ is asym. positive.



or



Exercise

$f(n) = \Theta(g(n))$ iff $g(n) = \Theta(f(n))$

Exercise

let $c > 0$. Prove that

(a) $c \cdot g(n) = O(g(n))$

(b) $c \cdot g(n) = \Omega(g(n))$

(c) $c \cdot g(n) = \Theta(g(n))$

Ex. $\sqrt{n+10} = \Theta(\sqrt{n})$ ✓

we must find pos. c_1, c_2, n_0 s.t.

$$0 \le c_1 \sqrt{n} \le \sqrt{n+10} \le c_2 \sqrt{n}$$

for all $n \ge n_0$.

Let $c_1 = 1$, $c_2 = \sqrt{2}$ and $n_0 = 10$.

Then, if $n \geq n_0$, we have

$$-10 \leq 0 \quad \text{and} \quad 10 \leq n$$

$$\therefore \; -10 \leq (1-1)n \quad \text{''} \quad 10 \leq (2-1)n$$

$$\therefore \; -10 \leq (1-c_1^2)n \quad \text{''} \quad 10 \leq (c_2^2-1)n$$

$$\therefore \; c_1^2 n \leq n+10 \quad \text{and} \quad n+10 \leq c_2^2 n$$

$$\therefore \; 0 \leq c_1^2 n \leq n+10 \leq c_2^2 n$$

$$\therefore \; 0 \leq c_1\sqrt{n} \leq \sqrt{n+10} \leq c_2\sqrt{n}. \qquad \blacksquare$$

<u>Exercise</u>: check that

$$c_1 = \sqrt{\tfrac{1}{2}}, \quad c_2 = \sqrt{\tfrac{3}{2}}, \quad n_0 = 20$$

also work.

Theorem

If $h(n) = O(g(n))$, and if $f(n) \le h(n)$ for all sufficiently large n, then $f(n) = O(g(n))$.

Proof.

by the hypotheses

- $\exists$. pos. c_1, n_1 s.t. $\forall n \ge n_1$:

$$0 \le h(n) \le c_1 \cdot g(n) \qquad (1)$$

- $\exists$ pos n_2 s.t. $\forall n \ge n_2$:

$$0 \le f(n) \le h(n) \qquad (2)$$

$\uparrow$

blanket assumption that $f(n)$ is a.n.n.

we must show that

- $\exists\ c_2, n_0$ pos. s.t. $\forall n \geq n_0$

$$0 \leq f(n) \leq c_2 \cdot g(n)$$

Let $c_2 = c_1$ and $n_0 = \max(n_1, n_2)$.

Then if $n \geq n_0$, we have both $n \geq n_1$ and $n \geq n_2$. Therefore

both

$$\begin{aligned} 0 \leq f(n) &\leq h(n) && \text{by } (2) \\ &\leq c_1 \cdot g(n) && \text{by } (1) \\ &= c_2 \cdot g(n) \end{aligned}$$

$\therefore\ 0 \leq f(n) \leq c_2\, g(n)$. ∎

<u>Exercise</u>

If $h(n) = \Omega(g(n))$ and $f(n) \geq h(n)$ for all suff. large n, then

$$f(n) = \Omega(g(n))$$

<u>Exercise</u>

If $h_1(n) = \Omega(g(n))$, $h_2(n) = O(g(n))$ and $h_1(n) \leq f(n) \leq h_2(n)$ for all suff. large n, then

$$f(n) = \Theta(g(n))$$

<u>Ex.</u>

Let $k \geq 0$ be fixed. Prove that

$$\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$$

<u>Proof.</u>

Observe

$$\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1} = O(n^{k+1})$$

Also

$$\sum_{i=1}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k$$

$$\geq \sum_{i \geq \lceil \frac{n}{2} \rceil}^{n} \lceil \tfrac{n}{2} \rceil^k$$

$$= \left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) \left\lceil \frac{n}{2} \right\rceil^k$$

$$= \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \left\lceil \frac{n}{2} \right\rceil^k \qquad \left\{ \begin{array}{l} \text{since} \\ \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = n \end{array} \right.$$

$$> \left(\frac{n}{2} - \cancel{1} + \cancel{1}\right) \left(\frac{n}{2}\right)^k \qquad \left\{ \begin{array}{l} \lfloor x \rfloor > x - 1 \\ \text{and} \\ \lceil x \rceil \geq x \end{array} \right.$$

$$= \left(\frac{n}{2}\right) \cdot \left(\frac{n}{2}\right)^k$$

$$= \left(\frac{1}{2}\right)^{k+1} \cdot n^{k+1} = \Omega\left(n^{k+1}\right)$$

we conclude that

$$\sum_{i=1}^{n} i^k = \Theta\left(n^{k+1}\right)$$

$\blacksquare$

This is consistent with

- $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \Theta(n^2)$$

- $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \Theta(n^3)$$

$\vdots$

<u>Exercise</u> $\underbrace{\sum_{i=1}^{n} \Theta(i)}_{\text{what does this mean?}} = \Theta(n^2)$

here $\Theta(i)$ stands for a fixed fcn. $f(i) = \Theta(i)$. so show

$$\sum_{i=1}^{n} f(i) = \Theta(n^2).$$

Goal: $\overset{\checkmark}{O}$, $\overset{\checkmark}{\Omega}$, $\overset{\checkmark}{\Theta}$, $\underline{\underline{o}}$, $\underline{\underline{\omega}}$

Defn

$$o(g(n)) = \{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) < c \cdot g(n) \}$$

Recall:

$$O(g(n)) = \{ f(n) \mid \exists c > 0, \exists n_0, \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n) \}$$

We see: $o(g(n)) \subseteq O(g(n))$.

Lemma 1:

$$f(n) = o(g(n)) \text{ iff } \lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = 0 .$$

Proof:

$f(n) = o(g(n))$ iff

$$\forall\, c>0, \exists\, n_0>0, \forall n \geq n_0 : 0 \leq \frac{f(n)}{g(n)} < c$$

this is the very definition of

$$\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = 0 \quad . \quad \blacksquare$$