CSE 102 11-30-21

Supplemental lecture 4

Problem

Given a graph $G=(V,E)$ with $n=|V|\geq 2$, determine whether or not G is connected.

We consider algorithms that only probe edges, i.e. ask only questions of the form: is $\{u,v\}\in E$ or is $\{u,v\}\notin E$.

We call these <u>adjacency questions</u>.

Adversary argument

Consider any algorithm for this problem and run it on an adversary that simulates such a graph.

<u>strategy</u>:

first partition V into two subsets X, Y of sizes $\lfloor \frac{n}{2} \rfloor$, $\lceil \frac{n}{2} \rceil$ respectively. Thus

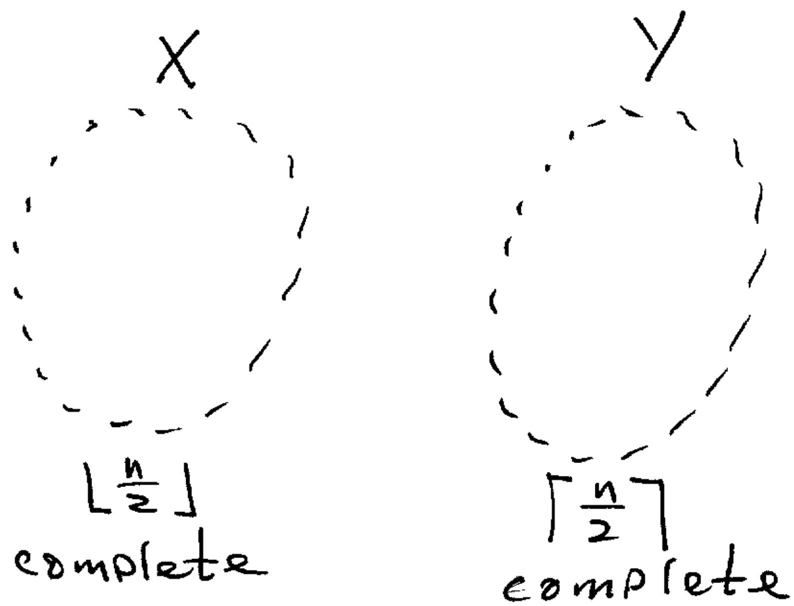
$$X \cup Y = V, \quad X \cap Y = \phi, \quad |X| = \lfloor \tfrac{n}{2} \rfloor, \quad |Y| = \lceil \tfrac{n}{2} \rceil .$$

when algorithm asks: "is u adj. to v?", then adversary answers yes iff u & v belong to the same set.

i.e.

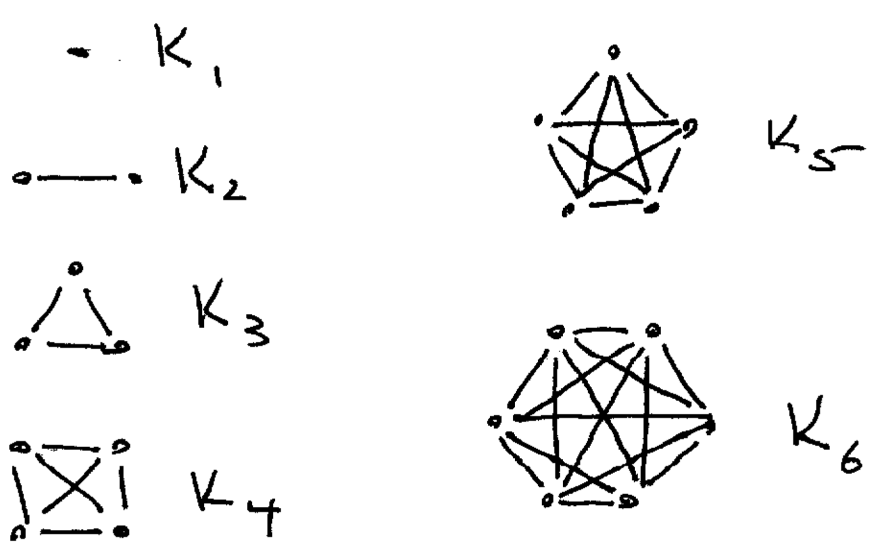
$$\text{answer} = \begin{cases} \text{yes} & \text{if } u,v \in X \ \underline{\text{or}}\ u,v \in Y \\ \text{no} & \text{if } u \in X, v \in Y \ \underline{\text{or}}\ u \in Y, v \in X \end{cases}$$

i.e. adversary answers as if G is the disjoint union of 2 complete graphs



<u>Aside</u> a graph is <u>complete</u> iff every pair of distinct vertices are joined by an edge.

<u>Ex</u>

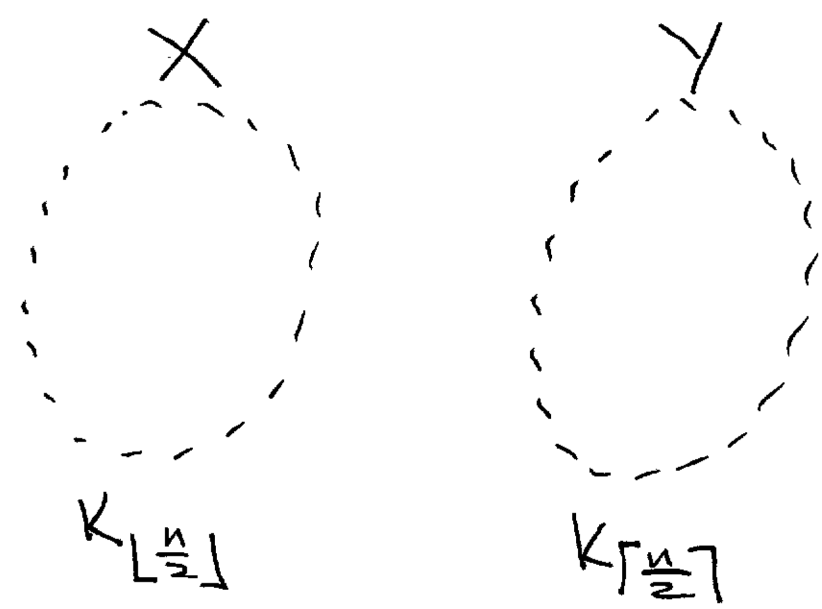


<u>Exercise</u> find formula for $|E(K_n)|$

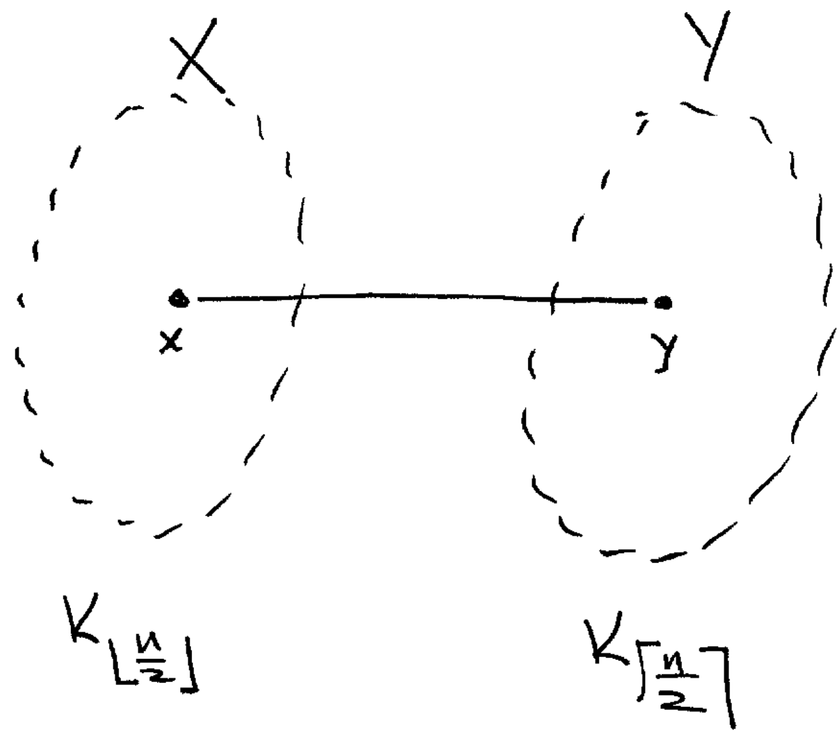
Suppose algorithm halts and returns an output (conn./disconn.) after doing fewer than $h(n) = \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$ edge probes. Then there must be a pair of vertices $x \in X$ and $y \in Y$ such that $\{x, y\}$ <u>has not</u> been probed.

Now, if algorithm says G is connected, the adversary can claim

$G =$



If algorithm says G is disconnected, adversary can claim

$G =$ 

In both cases, there is a graph that is both consistent with all adversary answers and which contradicts the algorithm's output.

$\therefore$ algorithm cannot be correct.

we conclude that $h(n) = \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$ is a lower bound on the number of edge probes that must be performed by a correct algorithm.

<u>Note</u>: $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil = \Omega(n^2)$

actually $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil \sim \frac{1}{4} n^2$

<u>Note</u>: Depth-First Search (DFS) solves problem in time $\Theta(n^2)$ (if graph is represented by adj matrix)

$\square$

<u>Note</u>: If G has n vertices then there are at most $|E(K_n)|$ edges in G. also note

$$|E(K_n)| = \#\text{ of } 2\text{-subsets of an } n\text{-set}$$
$$= \binom{n}{2}$$
$$= \frac{n(n-1)}{2} \sim \frac{1}{2}n^2$$

<u>Thm</u>
at least $\binom{n}{2}$ adjacency questions are necessary to determine whether a graph on n vertices is connected.

<u>Proof.</u>
consider any algorithm for this problem and run it <u>against</u> the following adversary.

strategy:

answer no to all edge probes, unless that answer would prove that G is disconnected.

More precisely: maintain two edge sets $A, B \subseteq E(K_n)$. Initially $B = \emptyset$ and $A = E(K_n)$. When an edge e is probed, do

Probe(e)

1. if $A - e$ is connected
2. $\quad A = A - e$
3. $\quad$ answer No
4. else
5. $\quad B = B + e$
6. $\quad$ answer yes

Identify A with graph $(V(K_n), A)$, and likewise B with $(V(K_n), B)$

observe that at all times $B \subseteq A$,
and $A - B = \{$edges not yet probed$\}$.
Also Both A, B are consistent with the entire seq. of adversary answers.

The following statements are true over any seq. of edge probes.

(a) A is always connected. obvious from the adversary strategy.

(b) If A contains a cycle, then none of its edges belong to B. [Proof: deleting a cycle edge from A, would leave A connected. such an edge could not have been added to B.]

(c) follows from (b) that B is acyclic.

(d) If $A \neq B$, then B is disconnected

[Proof: Assume, to get a $\times$, that B is connected, since $A \neq B$ and $B \subseteq A$, there exists $e \in A - B$. If we add e to B, then it would form a cycle in B. (B being acyclic and connected is a tree. By treeness thm, adding an edge creates a cycle.) Thus A contains a cycle that include some edges of B, contradicting (b).

$\therefore$ B must be disconnected.]

Suppose the algorithm halts and returns an answer (conn./disconn.) after doing fewer than

$$h(n) = \binom{n}{2}$$

edge probes. Then not every edge in $E(K_n)$ has been probed, $\therefore$ $A - B \neq \phi$, and hence $A \neq B$.

By (a) A is connected, and by (d) B is disconnected.

If algorithm says G is connected, adversary can claim $G = B$. If algorithm says G is disconnected, adversary can claim $G = A$.

So such an algorithm cannot be correct. Hence

$$h(n) = \binom{n}{2}$$

is a lower bound on the # of edge probes that must be performed.