CSE 102 11-10-22

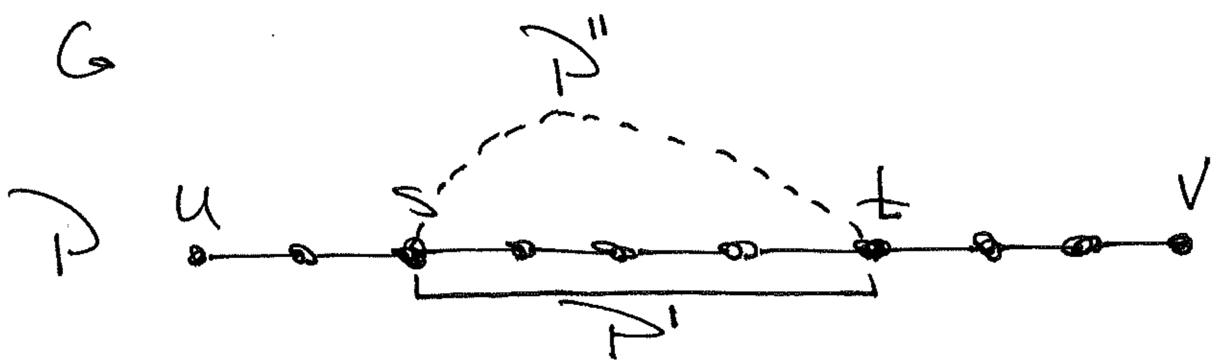
- Mid2 next Thursday 11-17-22

Lemma

Any sub-path of a shortest path in a graph G is also a shortest path.

Proof:

Let P be a shortest u-v path in G



Let s, t be intermediate vertices on P. Call sub-path of P from s to t P'. must show P' is a shortest s-t path. Assume, to get ⨳, that P' is not a shortest s-t path. Say P'' is a shortest s-t path:

$$\operatorname{len}(P'') < \operatorname{len}(P')$$

This gives us a u-v path in G, shorter than P.

This contradicts that P is a shortest u-v path. $\therefore$ no such s-t P'' can exist $\therefore$ P' is a shortest s-t path. ■

Problem

Matrix Chain multiplication

A matrix chain product is an expression

$$A_1 \cdot A_2 \cdots A_n$$

where each A_i is a matrix, of size : $p_{i-1} \times p_i \quad (1 \le i \le n)$

i.e.

$$\underset{(p_0 \times p_1)}{A_1} \cdot \underset{(p_1 \times p_2)}{A_2} \cdot \underset{(p_2 \times p_3)}{A_3} \; \cdots\cdots \; \underset{(p_{n-1} \times p_n)}{A_n}$$

input : $(p_0, p_1, \ldots, p_n)$

Say $n=3$: $\underset{p\times q}{A}\cdot\underset{q\times r}{B}\cdot\underset{r\times s}{C}$

<u>Cost</u> of $\underbrace{A\cdot B}_{p\times r}$: pr entries, each with q multiplications

so # mult in $A\cdot B$ is pqr

(1) $\overbrace{A}^{p\times q}\underset{\uparrow\; pqs}{\cdot}\left(\overbrace{B\underset{\uparrow\; qrs}{\cdot}C}^{q\times s}\right)$ has cost $= pqs + qrs$

(2) $\left(\overbrace{A\underset{\uparrow\; pqr}{\cdot}B}^{p\times r}\right)\underset{\uparrow\; prs}{\cdot}\overbrace{C}^{r\times s}$ has cost $= pqr + prs$

Ex. $p=10, q=100, r=10, s=100$

(1) $\rightarrow$ 200,000 mult.

(2) $\rightarrow$ 20,000 mult.

Problem: Given $A_1 \cdots A_n$ with sizes $p_0, p_1, \ldots, p_n$ find

- min # of mult.s necessary
- a parenthesization that achieves the min.

Note: The # of parenthesizations of $A_1 \cdots A_n$, denoted $P(n)$, satisfies

$$P(n) = \begin{cases} 1 & n = 1 \\ \displaystyle\sum_{k=1}^{n-1} P(k)P(n-k) & n \geq 2 \end{cases}$$

$$(A_1 \cdots A_k) \cdot (A_{k+1} \cdots A_n)$$

Exercise (hard)

$$P(n) = \frac{1}{n} \cdot \binom{2(n-1)}{n-1} = \Theta\left(\frac{4^n}{n^{3/2}}\right)$$

To define sub-problems, we consider sub-chains

$$A_i \cdots\cdots A_j$$

where $1 \le i \le j \le n$. Observe any such sub-instance has a top-level split point k

$$(A_i \cdots A_k) \cdot (A_{k+1} \cdots A_j)$$

where $i \le k < j$.

Lemma

If $A_i \cdots A_j$ is optimally parenthesized, then so are both factors $(A_i \cdots A_k)$ and $(A_{k+1} \cdots A_j)$.

Proof.

Assume that one of the factors is not opt. parenthesized, say left factor:

$$\overbrace{\underbrace{\underbrace{(A_i \cdots A_k)}_{\text{not opt.}}}_{p_{i-1} \times p_k} \cdot \underbrace{\underbrace{(A_{k+1} \cdots A_j)}_{\text{opt.}}}_{p_k \times p_j}}^{\text{opt.}}$$

This gives a parenthesization of $A_i \cdots A_j$ better than optimal! $\therefore$ $A_i \cdots A_k$ must be optimally parenthesized. $\blacksquare$

Let $m[i,j]$ be the min # of mult. necessary to compute

$$A_i \cdots A_j$$

for $1 \le i \le j \le n$

Goal: find $m[1,n]$

<u>note</u>: $m[i,i] = 0$

if $A_i \cdots A_j$ is opt. parenthesized, then

$$m[i,j] = m[i,k] + m[k+1,j] + p_{i-1}p_k p_j$$

since position k is unknown, we have

$$m[i,j] = \begin{cases} 0 & i = j \\ \min\limits_{i \le k < j} \left[m[i,k] + m[k+1,j] + p_{i-1}p_k p_j \right] & i < j \end{cases}$$

<u>note</u>: cell $m[i,j]$ depends on cells

$$m[i;\ i \cdots (j-1)]$$

and

$$m[(i+1) \cdots j;\ j]$$

see handout for examples.