- hw due tonight 10:00 pm
- reviews begin tomorrow
- hw2 posted

Defn

We say $f(n)$ is <u>asymptotically equivalent</u> to $g(n)$, and write

$$f(n) \sim g(n)$$

iff

$$\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = 1$$

Ex. $f(n) \sim g(n)$ iff $f(n) = g(n) + o(g(n))$

Proof:

$$f(n) \sim g(n) \text{ iff } \left(\frac{f(n)}{g(n)}\right) \to 1 \quad \text{as } n \to \infty$$

$$\text{iff } \left(\frac{f(n)}{g(n)} - 1\right) \to 0 \quad \text{"}$$

$$\text{iff } \frac{f(n) - g(n)}{g(n)} \to 0 \quad \text{"}$$

$$\text{iff } f(n) - g(n) = o(g(n))$$

$$\text{iff } f(n) = g(n) + o(g(n)) \quad \blacksquare$$

Handout on Common Functions

floors & ceilings

let $x \in \mathbb{R}$: $\lfloor x \rfloor$, $\lceil x \rceil$

Defn 1: $\lfloor x \rfloor$ = greatest int $\leq x$; $\lceil x \rceil$ = least int $\geq x$

Defn 2: $x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$

Defn 3: $N = \lfloor x \rfloor$ iff $N \leq x < N+1$ (1)

$N = \lceil x \rceil$ iff $N-1 < x \leq N$ (2)

Lemma 1: let $x \in \mathbb{R}$, $a, b \in \mathbb{Z}$. then

(1) $a \leq x < b$ iff $a \leq \lfloor x \rfloor < b$

(2) $a < x \leq b$ iff $a < \lceil x \rceil \leq b$

<u>lemma 2:</u> $x \in \mathbb{R}$, $m \in \mathbb{Z}^+$

(1) $\left\lfloor \frac{\lfloor x \rfloor}{m} \right\rfloor = \left\lfloor \frac{x}{m} \right\rfloor$

(2) $\left\lceil \frac{\lceil x \rceil}{m} \right\rceil = \left\lceil \frac{x}{m} \right\rceil$

<u>lemma 3:</u> $a, b, n \in \mathbb{Z}^+$

(1) $\left\lfloor \frac{\lfloor n/a \rfloor}{b} \right\rfloor = \left\lfloor \frac{n}{ab} \right\rfloor$

(2) $\left\lceil \frac{\lceil n/a \rceil}{b} \right\rceil = \left\lceil \frac{n}{ab} \right\rceil$

<u>Logarithms</u>

let $x, a, b \in \mathbb{R}$, $a > 1$, $b > 1$.

<u>defn</u> $\log_a(x) = \exp_a^{-1}(x)$

i.e.

$$x = a^{\log_a(x)} = \left(b^{\log_b(a)}\right)^{\log_a(x)}$$

$$= b^{\log_b(a)\cdot\log_a(x)}$$

$$\therefore \quad \log_b(x) = \underbrace{\log_b(a)}_{\text{const.}}\cdot\log_a(x) \qquad (*)$$

$$\therefore \quad \log_b(n) = \text{const}\cdot\log_a(n)$$

$$\therefore \quad \boxed{\log_b(n) = \Theta(\log_a(n))}$$

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

By eqn. $*$:

$$a^{\log_b(x)} = a^{\log_a(x)\cdot\log_b(a)}$$

$$= \left(a^{\log_a(x)}\right)^{\log_b(a)}$$

$$= x^{\log_b(a)}$$

$$\therefore \quad \boxed{a^{\log_b(x)} = x^{\log_b(a)}}$$

Stirling's formula

Let $n \in \mathbb{Z}^+$. Then

$$\boxed{n! = \sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)}$$

$$\left(\frac{n!}{\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n} - 1 \right) = \Theta\left(\frac{1}{n}\right)$$

Weaker version:

$$\boxed{n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n}$$

Corollary

(1) $n! = o(n^n)$

(2) $n! = \omega(b^n)$ for any $b > 1$

(3) $\log(n!) = \Theta(n \log n)$

Proof of (1)

$$\frac{n!}{n^n} = \frac{\sqrt{2\pi}\cdot\sqrt{n}\cdot\frac{\cancel{n^n}}{e^n}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)}{\cancel{n^n}}$$

$$= \underbrace{\sqrt{2\pi}}_{\downarrow \sqrt{2\pi}} \cdot \underbrace{\frac{n^{1/2}}{e^n}}_{\downarrow 0} \cdot \Big(\underbrace{1+\Theta\left(\tfrac{1}{n}\right)}_{\downarrow 1}\Big)$$

$$\therefore \quad \frac{n!}{n^n} \longrightarrow 0$$

$$\therefore \quad n! = o(n^n).$$

Proof of (3)

$$\frac{\log(n!)}{n\log n} = \frac{\log\left(\sqrt{2\pi}\cdot n^{1/2}\cdot \frac{n^n}{e^n}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)\right)}{n\log n}$$

$$= \frac{\log\sqrt{2\pi} + \frac{1}{2}\log n + n\log n - n\log e + \log\left(1+\Theta\left(\frac{1}{n}\right)\right)}{n\log n}$$

$$= \underbrace{\frac{\log\sqrt{2\pi}}{n\log n}}_{\downarrow\ 0} + \underbrace{\frac{1}{2n}}_{\downarrow\ 0} + \underbrace{1}_{\downarrow\ 1} - \underbrace{\frac{\log e}{\log n}}_{\downarrow\ 0} + \underbrace{\frac{\log\left(1+\Theta\left(\frac{1}{n}\right)\right)}{n\log n}}_{\downarrow\ 0}$$

$$\therefore \lim_{n\to\infty}\left(\frac{\log(n!)}{n\log n}\right) = 1$$

$$\therefore \log(n!) = \Theta(n\log n)$$

Exercise:

$$\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$$

where $\binom{m}{k} = \{\#$ of k-subsets of an m-set$\}$

$$= \frac{m!}{k!\,(m-k)!}$$

Proof:

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$

$$= \frac{\sqrt{2\pi \cdot 2n} \cdot \left(\frac{2n}{e}\right)^{2n} \cdot \left(1 + \Theta\left(\frac{1}{2n}\right)\right)}{\left(\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^{n} \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)\right)^2}$$

$$= \frac{\cancel{2}\sqrt{\pi}\cdot n^{1/2}\cdot \dfrac{2^{2n}\cdot \cancel{n^{2n}}}{\cancel{e^{2n}}}\cdot\left(1+\Theta\left(\frac{1}{2n}\right)\right)}{\cancel{2}\cdot\pi\cdot n\cdot \cancel{\dfrac{n^{2n}}{e^{2n}}}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)^2}$$

$$= \frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{n}}\cdot 4^n\cdot\left[\frac{1+\Theta\left(\frac{1}{2n}\right)}{\left(1+\Theta\left(\frac{1}{n}\right)\right)^2}\right]$$

$$\therefore \frac{\binom{2n}{n}}{\dfrac{4^n}{\sqrt{n}}} = \frac{1}{\sqrt{\pi}}\cdot\left[\quad\right] \xrightarrow[\text{as } n\to\infty]{} \frac{1}{\sqrt{\pi}}$$

Since $\frac{1}{\sqrt{\pi}} \in (0, \infty)$, we have

$$\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right).$$

$\blacksquare$