CSE 102 10-27-22

Exercise

count inversions

Inversions(A, p, r)

	cost:
if $p<r$	0
$q=\lfloor \frac{p+r}{2} \rfloor$	0
$a=$ Inversions(A,p,q)	$T(\lceil \frac{n}{2} \rceil)$
$b=$ Inversions$(A,q+1,r)$	$T(\lfloor \frac{n}{2} \rfloor)$
$c=$ Compare(A,p,q,r)	$\lceil \frac{n}{2} \rceil \cdot \lfloor \frac{n}{2} \rfloor$
return $a+b+c$	0
else	0
return 0	0

Compare(A, p, q, r)

```
count = 0
for i = p to q
    for j = q+1 to r
        if A[j] < A[i]      ← basic op.
            count++
return count
```

$$\#\text{ comparisons} = \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor$$

at top level: $p=1$, $r=n$, $q=\left\lfloor \frac{n+1}{2} \right\rfloor$

$$\text{length}(A[p..q]) = \left\lceil \frac{n}{2} \right\rceil$$

$$\text{length}(A[q+1...r]) = \left\lfloor \frac{n}{2} \right\rfloor$$

let $T(n) = \left(\begin{array}{l}\#\text{ comparisons by Inversions} \\ \text{on } A[1 \ldots n]\end{array}\right)$

so

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ T\left(\lfloor \frac{n}{2} \rfloor\right) + T\left(\lceil \frac{n}{2} \rceil\right) + \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil & \text{if } n \geq 2 \end{cases}$$

simplify

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n^2)$$

<u>compare</u>: n^2 to $n^{\log_2 2} = n$

$\therefore T(n) = \Theta(n^2)$

Exercise: figure out how to do this in $\Theta(n \log n)$ time.

Quicksort

Again $A[p \ldots r]$ is to be sorted.

Quicksort(A, p, r)	Worst case Cost
1. if $p < r$	0
2. $\quad q = \text{Partition}(A, p, r)$	$n-1$
3. $\quad \text{Quicksort}(A, p, q-1)$	$T(n-1)$
4. $\quad \text{Quicksort}(A, q+1, r)$	0

Partition(A, p, r) re-arranges $A[p \ldots r]$ and returns index q such that

$$\underbrace{A[p \ldots (q-1)]}_{\text{not sorted}} \le \underbrace{A[q]}_{} < \underbrace{A[(q+1) \ldots r]}_{\text{not sorted}}$$

Pivot element

q: Pivot index

<u>Exercise</u> Prove correctness of Quicksort() by induction on length($A[p \ldots r]$).

Assume correctness of Partition(A, p, r)

Partition(A, p, r)

1. $i = p-1$
2. for $j = p$ to $(r-1)$
3.   if $A[j] \le A[r]$
4. $i \mathrel{+}= 1$
5. $A[i] \leftrightarrow A[j]$ # swap
6. $A[i+1] \leftrightarrow A[r]$ # swap
7. return $(i+1)$

Invariants

$$\underbrace{A_p \cdots A_i}_{\le \text{Pivot}} \; \underbrace{A_{i+1} \cdots\cdots A_{j-1}}_{> \text{Pivot}} \; \underbrace{A_j \cdots\cdots A_{r-1}}_{\text{unknown}} \; \underbrace{A_r}_{\text{Pivot}}$$

If $A_j \le A_r$: swap $A_j \leftrightarrow A_{i+1}$ $i{+}{+}$, $j{+}{+}$

If $A_j > A_r$: $j{+}{+}$

Runtime

Partition

$\# \text{comp} = (r-p+1)-1 = r-p$

at top level

$\# \text{comp} = n-1$

Answers:

- worst case: $\Theta(n^2)$
- average case: $\Theta(n \log n)$.

<u>worst case</u>

occurs when $A[1 \cdots n]$ is already sorted. Partition gives

$$A[1 \cdots (n-1)] \le \underset{\substack{\text{Pivot} \\ q=n}}{A[n]} < \cdots \text{empty} \cdots$$

let $T(n)$ = worst case # comp. on $A[1 \cdots n]$.

so

$$T(n) = \begin{cases} 0 & \text{if } n = 0, 1 \\ T(n-1) + (n-1) & \text{if } n \ge 2 \end{cases}$$

By iteration:

$$T(n) = (n-1) + T(n-1)$$

$$= (n-1) + (n-2) + T(n-2)$$

$$= (n-1) + (n-2) + (n-3) + T(n-3)$$

$$\vdots$$

$$= \sum_{i=1}^{k} (n-i) + T(n-k)$$

halt when $n-k=1$ i.e. $\boxed{k=n-1}$

$$\therefore T(n) = \sum_{i=1}^{n-1} n - \sum_{i=1}^{n-1} i$$

$$= n(n-1) - \frac{n(n-1)}{2}$$

$$= \frac{1}{2}n(n-1) = \Theta(n^2).$$

Average case

Assume all $n!$ permutations of $A[1 \ldots n]$ are equally likely.

Let $t(n) = $ (avg. # of comparisons on $A[1 \ldots n]$).

$$t(n) = \frac{\displaystyle\sum_{\text{all permutations}} \begin{pmatrix}\text{\# of comp. by Q.S.}\\ \text{on given permutation}\end{pmatrix}}{n!}$$

<u>Goal</u>: • determine a recurrence for $t(n)$.

• solve it exactly

our assumption implies that q (returned from Partition) is equally likely to be any of $\{1, 2, 3, \ldots, n\}$.

$$\text{Prob}(q = i) = \frac{1}{n} \quad \text{for } i = 1, 2, \ldots, n.$$

<u>Recall</u> Partition does $n-1$ comp. on $A[1 \ldots n]$. also

$$\text{length}(A[1 \ldots (q-1)]) = q-1$$

and

$$\text{length}(A[(q+1) \ldots n]) = n-(q+1)+1$$
$$= n-q$$

Thus

$$t(n) = \frac{\sum_{q=1}^{n}\Big((n-1) + t(q-1) + t(n-q)\Big)}{n}$$

$$= \frac{1}{n}\cdot\sum_{q=1}^{n}(n-1) + \frac{1}{n}\cdot\sum_{q=1}^{n}\Big(t(q-1)+t(n-q)\Big)$$

$$= \frac{1}{\cancel{n}}\cdot\cancel{n}\cdot(n-1) + \frac{1}{n}\left(\sum_{q=2}^{n} t(q-1) + \sum_{q=1}^{n-1} t(n-q)\right)$$

(<u>note</u>: $t(0)=0,\ t(1)=0$)

$$= (n-1) + \frac{1}{n}\left(\sum_{q=1}^{n-1} t(q) + \sum_{q=1}^{n-1} t(n-q)\right)$$

$$\therefore\quad t(n) = (n-1) + \frac{2}{n}\cdot\sum_{q=1}^{n-1} t(q)$$

Let

$$x_n = \sum_{q=1}^{n-1} t(q) \quad , \quad x_1 = 0$$

Then

$$x_{n+1} - x_n = \sum_{q=1}^{n} t(q) - \sum_{q=1}^{n-1} t(q) = t(n)$$

so

$$x_{n+1} - x_n = (n-1) + \frac{2}{n} \cdot x_n$$

$$\therefore \quad x_{n+1} - \left(\frac{n+2}{n}\right) x_n = n-1$$

multiply by : $\dfrac{1}{(n+1)(n+2)}$

$$\frac{x_{n+1}}{(n+1)(n+2)} - \frac{x_n}{n(n+1)} = \frac{n-1}{(n+1)(n+2)}$$

$$\therefore \quad \frac{x_{n+1}}{(n+1)(n+2)} - \frac{x_n}{n(n+1)} = \frac{3}{n+2} - \frac{2}{n+1}$$

replace n by k:

$$\frac{x_{k+1}}{(k+1)(k+2)} - \frac{x_k}{k(k+1)} = \frac{3}{k+2} - \frac{2}{k+1}$$

sum for $k=1$ to $n-1$:

$$\sum_{k=1}^{n-1} \left(\frac{x_{k+1}}{(k+1)(k+2)} - \frac{x_k}{k(k+1)} \right) = \sum_{k=1}^{n-1} \left(\frac{1}{k+2} + \frac{2}{k+2} - \frac{2}{k+1} \right)$$

$$\therefore \sum_{k=1}^{n-1} \frac{x_{k+1}}{(k+1)(k+2)} - \sum_{k=1}^{n-1} \frac{x_k}{k(k+1)} = \sum_{k=3}^{n+1} \frac{1}{k} + \sum_{k=1}^{n-1} \frac{2}{k+2} - \sum_{k=1}^{n-1} \frac{2}{k+1}$$

$$\frac{x_n}{n(n+1)} - \cancelto{0}{\frac{x_1}{2}} = \sum_{k=3}^{n+1} \frac{1}{k} + \frac{2}{n+1} - 1$$

$$\frac{x_n}{n(n+1)} = \left(\sum_{k=1}^{n} \frac{1}{k} \right) - 1 - \frac{1}{2} + \frac{1}{n+1} + \frac{2}{n+1} - 1$$

$$= \sum_{k=1}^{n} \frac{1}{k} - \frac{5}{2} + \frac{3}{n+1}$$

Define $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Then

$$\frac{x_n}{n(n+1)} = H_n - \frac{5}{2} + \frac{3}{n+1}$$

$$\therefore \quad x_n = \underline{3n - \frac{5}{2} n(n+1) + n(n+1) H_n}$$

Recall

$$t(n) = (n-1) + \frac{2}{n} \cdot x_n$$