CSE 102 10-18-22

- hw3 due tonite
  revised. ~~#7~~.
- midterm Thursday.

Ex.

$$T(n) = \begin{cases} 5 & \text{if } n = 0, 1 \\ T(n-2) + n & \text{if } n \geq 2 \end{cases}$$

n	0	1	2	3	4	...
$T(n)$	5,	5,	7,	8,	11,	...

Iteration:

$$T(n) = n + T(n-2)$$
$$= n + \Big((n-2) + T\big((n-2)-2\big)\Big)$$

$$= n + (n-2) + T(n-4)$$

$$= n + (n-2) + (n-4) + T(n-6)$$

$$= n + (n-2) + (n-4) + (n-6) + T(n-8)$$

$$\vdots$$

$$= \sum_{i=0}^{k-1} (n-2i) + T(n-2k)$$

recurrence terminates when

$$0 \le n - 2k < 2$$

$$2k \le n < 2k + 2$$

$$k \le \frac{n}{2} < k + 1$$

$$\therefore \boxed{k = \left\lfloor \frac{n}{2} \right\rfloor}.$$

for this k: $T(n-2k) = 5$

$$\therefore$$

$$T(n) = \sum_{i=0}^{\overset{\lfloor \frac{n}{2} \rfloor}{\downarrow}\\ k-1} (n-2i) + 5$$

$$= n \cdot \sum_{i=0}^{k-1} 1 - 2 \sum_{i=\cancel{0}1}^{k-1} i + 5$$

$$= nk - 2 \cdot \frac{k(k-1)}{2} + 5$$

$$= nk - k(k-1) + 5$$

$$\therefore \quad T(n) = n\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + 5$$

$$\sim \frac{1}{2}n^2 - \frac{1}{4}n^2 + \frac{1}{2}n + 5$$

$$\therefore \quad T(n) = \Theta(n^2).$$

Ex.

$$T(n) = \begin{cases} 0 & n = 1 \\ T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1 & n \geq 2 \end{cases}$$

$$T(n) = 1 + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$$

$$T(\text{☺}) = 1 + T\left(\left\lfloor \frac{\text{☺}}{2} \right\rfloor\right)$$

$$\begin{aligned} T(n) &= 1 + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \\ &= 1 + 1 + T\left(\left\lfloor \frac{\lfloor n/2 \rfloor}{2} \right\rfloor\right) \\ &= 2 + T\left(\left\lfloor \frac{n}{2^2} \right\rfloor\right) \\ &= 2 + 1 + T\left(\left\lfloor \frac{\lfloor n/2^2 \rfloor}{2} \right\rfloor\right) \\ &= 3 + T\left(\left\lfloor \frac{n}{2^3} \right\rfloor\right) \\ &\;\vdots \\ &= K + T\left(\left\lfloor \frac{n}{2^K} \right\rfloor\right) \end{aligned}$$

terminate when

$$1 \le \left\lfloor \frac{n}{2^k} \right\rfloor < 2$$

$$1 \le \frac{n}{2^k} < 2$$

$$2^k \le n < 2^{k+1}$$

$$k \le \lg(n) < k+1$$

$$\therefore \quad k = \lfloor \lg(n) \rfloor$$

$$\therefore \quad T(n) = \lfloor \lg(n) \rfloor$$

$$\therefore \quad T(n) = \Theta(\log n).$$

Exercise

$$S(n) = \begin{cases} 0 & n = 1 \\ S(\lceil \frac{n}{2} \rceil) + 1 & n \geq 2 \end{cases}$$

show $S(n) = \lceil \lg(n) \rceil$, hence

$S(n) = \Theta(\log n)$.

Ex.

$$T(n) = \begin{cases} c & 1 \leq n < n_0 \\ T(\lfloor \frac{n}{2} \rfloor) + d & n \geq n_0 \end{cases}$$

$$\begin{aligned} T(n) &= d + T(\lfloor \tfrac{n}{2} \rfloor) \\ &= d + d + T\left(\left\lfloor \frac{\lfloor n/2 \rfloor}{2} \right\rfloor\right) \\ &= 2d + T(\lfloor \tfrac{n}{2^2} \rfloor) \\ &= 2d + d + T\left(\left\lfloor \frac{\lfloor n/2^2 \rfloor}{2} \right\rfloor\right) \\ &= 3d + T(\lfloor \tfrac{n}{2^3} \rfloor) \end{aligned}$$

$$\vdots$$

$$= kd + T\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right)$$

end when $1 \le \left\lfloor \frac{n}{2^k} \right\rfloor < n_0$.

$$\therefore \quad 1 \le \frac{n}{2^k} < n_0$$

$$\therefore \quad \frac{n}{n_0} < 2^k$$

$$\therefore \quad \lg\left(\frac{n}{n_0}\right) < k$$

We seek smallest such k, so

$$k-1 \le \lg\left(\frac{n}{n_0}\right) < k$$

$$\therefore \quad k-1 = \left\lfloor \lg\left(\frac{n}{n_0}\right) \right\rfloor$$

$$\therefore \quad k = \left\lfloor \lg\left(\frac{n}{n_0}\right) \right\rfloor + 1$$

Hence

$$T(n) = \left(\left\lfloor \lg\left(\frac{n}{n_0}\right)\right\rfloor + 1\right) d + e$$

$$\therefore \quad T(n) = \Theta(\log n)$$

Master method

find asymptotic solutions to

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Require $a \geq 1$, $b > 1$, and $f(n)$ is asymptotically positive.

Master Theorem

Given above, we have 3 cases:

(1) if $f(n) = O(n^{\log_b(a) - \varepsilon})$ for some $\varepsilon > 0$, then

$$\boxed{T(n) = \Theta(n^{\log_b(a)})}$$

(2) if $f(n) = \Theta(n^{\log_b(a)})$, then

$$\boxed{T(n) = \Theta(f(n) \cdot \log(n))}$$

(3) if $f(n) = \Omega(n^{\log_b(a) + \varepsilon})$ for some $\varepsilon > 0$, and $\boxed{\text{if } a f(\tfrac{n}{b}) \leq c f(n) \text{ for some } 0 < c < 1 \text{ and all suff. large } n}$, then

$$\boxed{T(n) = \Theta(f(n))}$$

regularity condition

Remarks:

- in all cases compare

$$f(n) \text{ to } n^{\log_b(a)}$$

<u>case 1</u>: $n^{\log_b(a)}$ wins by the polynomial factor n^{ε}

<u>case 2</u>: tie.

<u>case 3</u>: $f(n)$ wins by the polynomial factor n^{ε}

- conclusion depends on $\Theta(f(n))$ only. sometimes write

$$T(n) = aT\left(\left\lfloor \frac{n}{b} \right\rfloor\right) + \Theta(f(n))$$

Ex. $T(n) = 8T\left(\frac{n}{2}\right) + n^3$

compare: n^3 to $n^{\log_2 8} = n^3$

By case 2:

$$T(n) = \Theta(n^3 \log n).$$

Ex. $T(n) = 5T\left(\frac{n}{4}\right) + n$

compare: n^1 to $n^{\log_4(5)}$

Pick $\varepsilon = \log_4(5) - 1 > 0$ since $\log_4 5 > 1$.

Then $1 = \log_4(5) - \varepsilon$, and

$$n = O(n^1) = O\left(n^{\log_4(5) - \varepsilon}\right).$$

by case 1:

$$T(n) = \Theta\left(n^{\log_4(5)}\right).$$

Ex. $T(n) = 5T\left(\frac{n}{4}\right) + n^2$

compare: n^2 to $n^{\log_4(5)}$

$5 < 16 \Rightarrow \log_4(5) < \log_4(16) = 2$. let

$\boxed{\varepsilon = 2 - \log_4(5)}$. $\therefore \varepsilon > 0$. Also

$2 = \log_4(5) + \varepsilon$, and hence

$$n^2 = \Omega(n^2) = \Omega\left(n^{\log_4(5)+\varepsilon}\right)$$

need c in range $0 < c < 1$ s.t.

$$5\left(\frac{n}{4}\right)^2 \leq cn^2$$

i.e. $\frac{5}{16} n^2 \leq cn^2$

i.e. $\frac{5}{16} \leq c$

Possible since $\frac{5}{16} < 1$, since $\underline{5 < 16}$

$\boxed{\text{Pick any } c \text{ in } \frac{5}{16} \leq c < 1}$.

By case 3

$$\boxed{T(n) = \Theta(n^2)}$$

Exercise

let $T(n) = aT\left(\frac{n}{b}\right) + n^d$

where $d > \log_b(a)$. show

- case 3 applies!

  $$n^d = \Omega\left(n^{\log_b a - \varepsilon}\right)$$

- regularity cond. holds.

$\therefore f(n) = \Theta(n^d)$

Ex.

$$T(n) = T(\lfloor \tfrac{n}{2} \rfloor) + T(\lceil \tfrac{n}{2} \rceil) + \log(n!)$$

simplify:

$$T(n) = 2T\left(\frac{n}{2}\right) + n\log n$$

compare $n\log(n)$ to $n^{\log_2 2} = n$

Master thm does not apply,

need $n\log n = \Omega(n^{1+\varepsilon})$, some $\varepsilon > 0$.

no such $\varepsilon > 0$ exists! why?

$$\frac{n\log n}{n^{1+\varepsilon}} = \frac{\cancel{n}\log n}{\cancel{n}\cdot n^{\varepsilon}} \longrightarrow 0 \quad \text{as } n \to \infty$$

$\therefore$ $n\log n = o(n^{1+\varepsilon})$, and $o(n^{1+\varepsilon}) \cap \Omega(n^{1+\varepsilon}) = \phi$

$\therefore$ $n \log n \neq \Omega(n^{1+\varepsilon})$ for <u>any</u> $\varepsilon > 0$. $\therefore$ Case 3 does not apply, so master thm doesn't apply.