CSE 102 10-13-22

- midterm 1 : Thur 10-20-22
  lecture to follow

Recurrence Relations

Ex.

$$T(n) = \begin{cases} c & n = 1 \\ T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n}{2} \right\rceil\right) + dn & n > 1 \end{cases}$$

methods

- Substitution:
- Recursion tree - Iteration
- Master theorem

Ex

$$T(n) = \begin{cases} 2 & 1 \le n < 3 \\ 3T(\lfloor \frac{n}{3} \rfloor) + n & n \ge 3 \end{cases}$$

Guess: $T(n) = O(n \log(n))$

To prove this we find Pos. c, n_0

s.t.

$$\forall n \ge n_0 : \boxed{T(n) \le c n \log n}$$

base: $n=1$, $n=2$ in recurrence

note: if $n=1$ ~~false~~ !!

$$T(1) = 2$$

but $c \cdot 1 \cdot \log(1) = 0$

and $2 \not\leq 0$.

Base cases $n = 2, 3, 4, 5, \ldots$?

$$T(2) = 2 \overset{\text{need}}{\leq} c \cdot 2 \log 2$$

$$T(3) = 3 \cdot 2 + 3 = 9 \leq c\, 3 \log 3$$

$$T(4) = 3 \cdot 2 + 4 = 10 \leq c\, 4 \log 4$$

$$T(5) = 3 \cdot 2 + 5 = 11 \leq c\, 5 \log 5$$

$\vdots$

lowest base case is $\boxed{n_0 = 2}$

highest base case $\boxed{n_1}$

mimic induction step. let $\log = \log_3$

let $n > n_1$. Assume for all

$k \geq 2$ that $T(k) \leq c k \log k$.
(and $k < n$)

We must show: $T(n) \leq c n \log n$.

Then

$$T(n) = 3T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n$$

$$\leq 3 \cdot c \left\lfloor \frac{n}{3} \right\rfloor \cdot \log \left\lfloor \frac{n}{3} \right\rfloor + n$$

by ind hyp
with $k = \lfloor n/3 \rfloor$
and
$2 \leq \left\lfloor \frac{n}{3} \right\rfloor < n$

$$\leq \cancel{3} \cdot c \left(\frac{n}{\cancel{3}}\right) \cdot \log \left(\frac{n}{3}\right) + n$$

since $\lfloor x \rfloor \leq x$

$$= cn(\log n - \log 3) + n$$

$$= cn \log n - cn + n \leq cn \log n$$

want

want $-cn + n \le 0$

$$n \le cn$$

$$\boxed{1 \le c}$$ given this ind. step works

also need:

$$c \ge \max\left(3, \frac{10}{4\log 4}, \frac{11}{5\log 5}, \frac{1}{\log 2}\right)$$ given this, base cases work.

let

$$c = \max\left(1, \frac{1}{\log 2}, 3, \frac{10}{4\log 4}, \frac{11}{5\log 5}\right) = 3$$

have $\boxed{c = 3,\ n_0 = 2,\ n_1 = 5}$

<u>claim</u> : $\forall n \geq 2 : \boxed{T(n) \leq 3n\log_3(n)}$

<u>Proof.</u>

I. $T(n) \leq 3n \log n$ in cases $n = 2, 3, 4, 5$

Becomes

$2 \leq 6 \log 2$ ✓ check

$9 \leq 9$ ✓

$10 \leq 12 \log 4$ ✓

$11 \leq 15 \log 5$ ✓

II. $\forall n > 5 : \left(P(2) \wedge \ldots \wedge P(n-1)\right) \to P(n).$

let $n > 5$. assume for all k in range $2 \leq k < n$ that

$$T(k) \leq 3k \log k.$$

we must show

$$T(n) \leq 3n\log n \checkmark$$

so

$$T(n) = 3T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n$$

$$\leq 3 \cdot 3\left\lfloor \frac{n}{3} \right\rfloor \log\left\lfloor \frac{n}{3} \right\rfloor + n \quad \left\{ \begin{array}{l} \text{by ind.} \\ \text{hyp and} \\ 2 \leq \left\lfloor \frac{n}{3} \right\rfloor \end{array} \right.$$

$$\leq 9\left(\frac{n}{3}\right)\log\left(\frac{n}{3}\right) + n \quad \left\{ \begin{array}{l} \text{since} \\ \lfloor x \rfloor \leq x \end{array} \right.$$

$$= 3n(\log n - 1) + n$$

$$= 3n\log n - 3n + n$$

$$= 3n\log n - 2n$$

$$\leq 3n\log n \, .$$

$\blacksquare$

Ex. recursion tree

$$T(n) = \begin{cases} 1 & 1 \le n < 3 \\ 2T(\lfloor \frac{n}{3} \rfloor) + n & n \ge 3 \end{cases}$$

Write $T(n) = 2T\left(\frac{n}{3}\right) + n$

write $\underbrace{T(n) = n + T\left(\frac{n}{3}\right) + T\left(\frac{n}{3}\right)}$

generates trees

Write: $T(n) = n + 2T\left(\frac{n}{3}\right)$

$$= n + 2\left(\frac{n}{3} + 2T\left(\frac{n}{3^2}\right)\right)$$

$$= n + 2 \cdot \frac{n}{3} + 2^2 \cdot T\left(\frac{n}{3^2}\right)$$

$$= n + 2 \cdot \frac{n}{3} + 2^2\left(\frac{n}{3^2} + 2 \cdot T\left(\frac{n}{3^3}\right)\right)$$

$$= n + 2 \cdot \frac{n}{3} + 2^2 \cdot \frac{n}{3^2} + 2^3 \cdot T\left(\frac{n}{3^3}\right)$$

$\vdots$

$$= \left[n + 2\cdot\frac{n}{3} + 2^2\cdot\frac{n}{3^2} + \cdots + 2^{k-1}\cdot\frac{n}{3^{k-1}}\right] + 2^k T\left(\frac{n}{3^k}\right)$$

$$\therefore T(n) = \sum_{i=0}^{k-1} 2^i\cdot\left(\frac{n}{3^i}\right) + 2^k T\left(\frac{n}{3^k}\right)$$

$$T(n) = n\cdot\sum_{i=0}^{k-1}\left(\frac{2}{3}\right)^i + 2^k T\left(\frac{n}{3^k}\right)$$

Find k st. $\frac{n}{3^k} = 1$, i.e.

$$n = 3^k$$

$$\boxed{\log_3(n) = k}$$

so

$$T(n) = n\cdot\sum_{i=0}^{k-1}\left(\frac{2}{3}\right)^i + 2^{\log_3(n)}\cdot T(1)$$

$$= n\cdot\frac{1-\left(\frac{2}{3}\right)^k}{1-\left(\frac{2}{3}\right)} + n^{\log_3(2)}\cdot 1$$

$$= 3n\left(1-\left(\frac{2}{3}\right)^k\right) + n^{\log_3(2)}$$

so $T(n) \le 3n + n^{\log_3(2)}$

Guess $T(n) \le O(n)$

Iteration method

write

$$T(n) = n + 2T\left(\left\lfloor \frac{n}{3} \right\rfloor\right)$$

$$= n + 2\left(\left\lfloor \frac{n}{3} \right\rfloor + 2T\left(\left\lfloor \frac{\lfloor n/3 \rfloor}{3} \right\rfloor\right)\right)$$

$$= n + 2\left\lfloor \frac{n}{3} \right\rfloor + 2^2 \cdot T\left(\left\lfloor \frac{n}{3^2} \right\rfloor\right)$$

$$= n + 2\left\lfloor \frac{n}{3} \right\rfloor + 2^2 \cdot \left(\left\lfloor \frac{n}{3^2} \right\rfloor + 2T\left(\left\lfloor \frac{\lfloor n/3^2 \rfloor}{3} \right\rfloor\right)\right)$$

$$= n + 2\left\lfloor \frac{n}{3} \right\rfloor + 2^2\left\lfloor \frac{n}{3^2} \right\rfloor + 2^3 \cdot T\left(\left\lfloor \frac{n}{3^3} \right\rfloor\right)$$

$$\vdots$$

$$\therefore T(n) = \sum_{i=0}^{k-1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor + 2^k \cdot T\left(\left\lfloor \frac{n}{3^k} \right\rfloor\right)$$

recursion terminates when

$$1 \le \left\lfloor \frac{n}{3^k} \right\rfloor < 3$$

$$\therefore \quad 1 \le \frac{n}{3^k} < 3$$

$$3^k \le n < 3^{k+1}$$

$$k \le \log_3(n) < k+1$$

$$\therefore \quad \boxed{k = \lfloor \log_3(n) \rfloor}$$

Hence

$$T(n) = \sum_{i=0}^{k-1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor + 2^k \cdot 1$$

estimate upward:

$$T(n) = \sum_{i=0}^{k-1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor + 2^{\lfloor \log_3(n) \rfloor}$$

$$\le n \sum_{i=0}^{k-1} \left(\frac{2}{3}\right)^i + 2^{\log_3 n}$$

$$= n \left(\frac{1-(2/3)^k}{1-(2/3)} \right) + n^{\log_3 2}$$

$$= 3n\left(1-\left(\tfrac{2}{3}\right)^k\right) + n^{\log_3 2}$$

$$\le 3n + n^{\log_3 2} = O(n)$$

$$\therefore \boxed{T(n) = O(n)}$$