cse 102 10-11-22

Ex define $T(n)$ for $n \in \mathbb{Z}^+$:

$$T(n) = \begin{cases} 1 & \text{if } 1 \le n \le 2 \\ 4T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n & \text{if } n \ge 3 \end{cases}$$

show $\forall n \ge 1$: $\boxed{T(n) \le n^2}$ hence $T(n) = O(n^2)$.

$P(n)$

strong ind., 2 base cases: $n=1$, $n=2$.

I. $T(1) = 1 \le 1 = 1^2$ ✓

$T(2) = 1 \le 4 = 2^2$ ✓

II. $\forall n \geq 2 : P(1) \wedge \dots \wedge P(n-1) \rightarrow P(n)$

Let $n \geq 2$. assume for all k in the range $1 \leq k < n$ that

$$T(k) \leq k^2 .$$

we must show

$$T(n) \leq n^2 .$$

Then

$$T(n) = 4T(\lfloor \tfrac{n}{3} \rfloor) + n$$

$$\leq 4 \lfloor \tfrac{n}{3} \rfloor^2 + n \quad \left\{ \begin{array}{l} \text{by the ind hyp} \\ \text{with } k = \lfloor \tfrac{n}{3} \rfloor, \\ \text{since } 1 \leq \lfloor n/3 \rfloor < n . \end{array} \right.$$

$$\leq 4 \left(\tfrac{n}{3}\right)^2 + n \quad \{ \text{since } \lfloor x \rfloor \leq x$$

$$= \tfrac{4}{9} n^2 + n \leq n^2$$

why?

$$n > 2 \iff \frac{9}{5} \le n$$

$$\iff n \le \frac{5}{9} n^2$$

$$\iff \frac{4}{9} n^2 + n \le n^2 \quad \checkmark$$

$$\therefore \quad T(n) \le n^2 . \qquad \blacksquare$$

n	1	2	3	4	5	6	7	8	9
$\lfloor n/3 \rfloor$	·	·	1	1	1	2	2	2	3

$\uparrow \quad \uparrow$

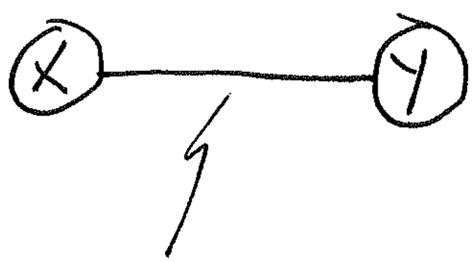
Base cases.

Graph Theory

Defn a graph $G = (V, E)$ is a pair of sets

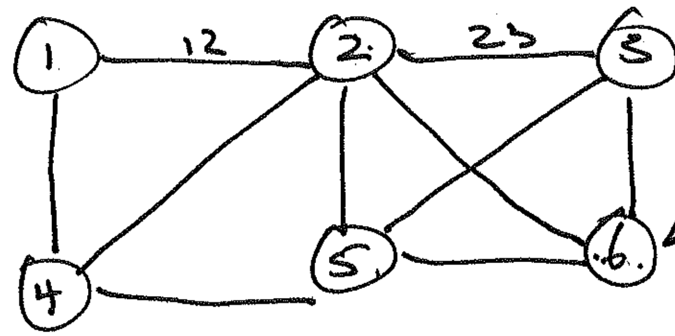
Vertices: $V \neq \emptyset$ vertices

Edges: $E \subseteq V^{(2)} = \{\text{2-element subsets of } V\}$



$\{x, y\} = xy = yx$

Ex.



$\deg(6) = 3$

$V = \{1, 2, 3, 4, 5, 6\}$

$E = \{12, 14, 23, 24, 25, 26, 35, 36, 45, 56\}$

We say: 2 is adjacent to 3

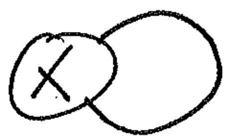
23 joins 2 to 3

23 has ends 2 and 3

12 is adjacent to 23

2 is incident with 23

note: not allowed



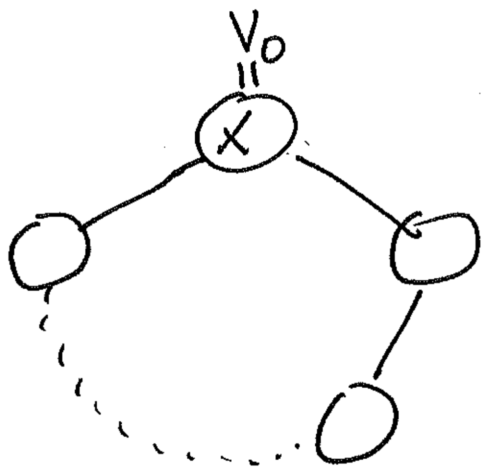
let $x, y \in V$. An x-y path in G is a sequence

$$x = v_0, v_1, v_2, \ldots, v_k = y$$

of vertices in which $\{v_i, v_{i+1}\} \in E(G)$ for all $i = 0, 1, \ldots, k-1$. Also there are no repeated vertices (except possibly when $x = y$.) if $x = y$ the path is closed.

trivial path: $x = v_0 = y$

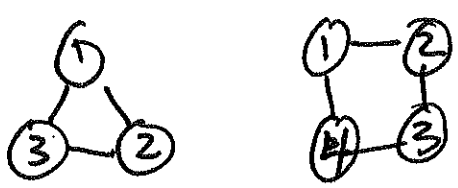
a non-trivial closed path



is called a <u>cycle</u>.

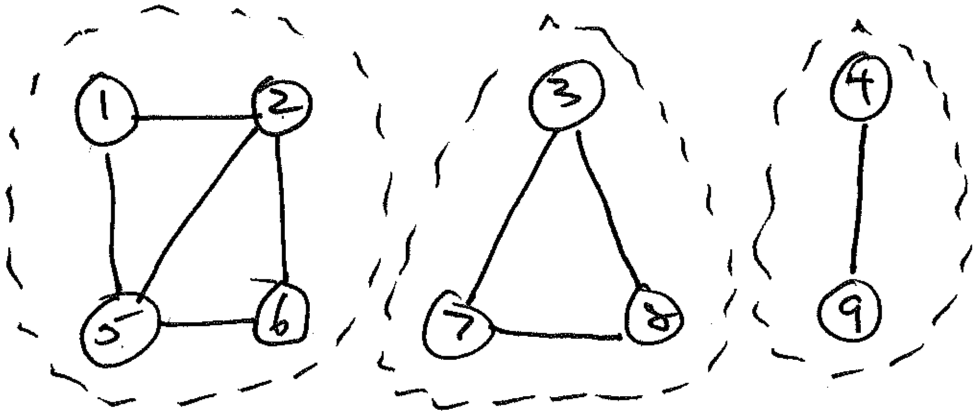
length of path = # edges traversed.

<u>Cycles</u>:



len = 3 , 4 , 5 , 6 . . . .

G is called <u>connected</u> if for all $x, y \in V$, G contains an x-y path.

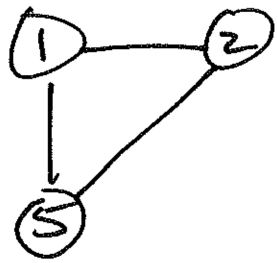
G 

not connected

A <u>subgraph</u> of G is a graph H such that

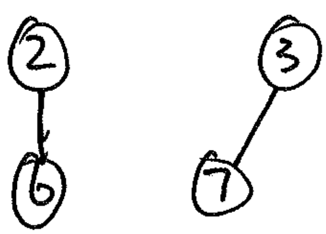
- $V(H) \subseteq V(G)$
- $E(H) \subseteq E(G)$

<u>Note</u>: $(\{1,2,5\}, \{12, 15, 25\})$



is a connected subgraph of G.

<u>Also</u>: $(\{2,3,6,7\}, \{26, 37\})$



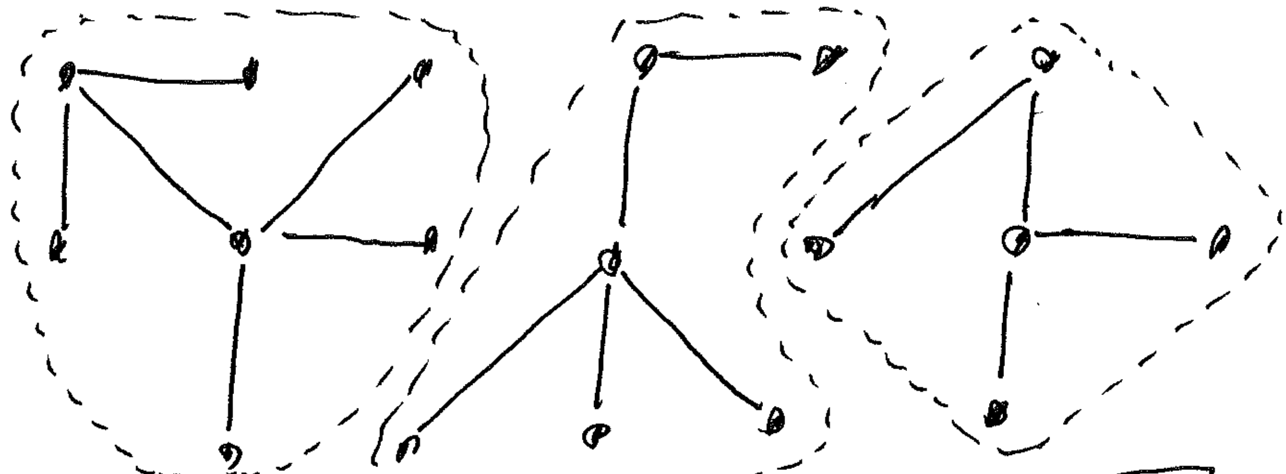
is a disconnected subgraph of G.

A subgraph H of G is a <u>connected component</u> iff

(1) H is connected

(2) H is maximal w.r.t. (1)

G is called <u>acyclic</u> iff it contains no cycles. (also called a <u>forest</u>.)



#vert:	7	6	5
#edge:	6	5	4

A graph is a <u>tree</u> iff it is both connected and acyclic.

Ex. Prove $\forall n \geq 1$: $\boxed{\text{if } T \text{ is a tree on } n \text{ vertices, then } T \text{ has } (n-1) \text{ edges.}}$

$$\forall n \geq 1 : \boxed{|V(T)| = n \Rightarrow |E(T)| = n-1}$$

$P(n)$

Proof:

I. let T be a tree with 1 vertex.

Then $T = (\{1\}, \emptyset)$.

1

In this case T has $1-1=0$ edges.

IId. $\forall n > 1 : P(1) \wedge \ldots \wedge P(n-1) \rightarrow P(n)$

Let $n > 1$ be arbitrary. Assume for all k in the range $1 \leq k < n$ that: if T' is a tree on k vertices, then T' has $(k-1)$ edges.

we must show that: if T is a tree on n vertices, then T has $(n-1)$ edges.

Let T be a tree on n vertices. Pick any edge $e \in E(T)$ and remove it from T. This disconnects T into 2 subtrees, call them T_1 and T_2.

let $|V(T_1)| = k_1$, and $|V(T_2)| = k_2$

note $1 \le k_1 < n$ and $1 \le k_2 < n$

By the ind. hyp. we have

$$|E(T_1)| = k_1 - 1 \text{ and } |E(T_2)| = k_2 - 1,$$

note also that $k_1 + k_2 = n$, since no vertices were removed. Therefore

$$\begin{aligned} |E(T)| &= |E(T_1)| + |E(T_2)| + 1 \\ &= (k_1 - 1) + (k_2 \cancel{- 1}) \cancel{+ 1} \\ &= (k_1 + k_2) - 1 \\ &= n - 1. \end{aligned}$$

$\blacksquare$

Induction Principles

Theorem 1

for any Prop. fcn $P(n)$, the following sentence is true

$$\left[P(1) \wedge \left(\forall n > 1 : P(n-1) \rightarrow P(n)\right)\right] \rightarrow \left[\forall n \geq 1 : P(n)\right]$$

will use: well ordering Property $\mathbb{Z}^+$.

any non-empty subset of $\mathbb{Z}^+$ contains a least element.

Proof of Thm1

Assume $P(1)$ and $\forall n>1: P(n-1) \rightarrow P(n)$ are both true. Define

$$S = \{ n \in \mathbb{Z}^+ \mid P(n) \text{ is false} \}$$

it's sufficient to show $S = \phi$.

Assume, to get a 'X', that $S \neq \phi$.

By the W.O.P, S contains a least element, call it m.

Since $P(1)$ is true, $1 \notin S$, so $m \neq 1$. $\therefore m > 1$, and $m-1$ is a positive int.

note $m-1 \notin S$, so $\boxed{P(m-1) \text{ is true.}}$ we have $\forall n>1: P(n-1) \Rightarrow P(n)$

$\therefore$ letting $n=m$ we have

$$\boxed{P(m-1) \Rightarrow P(m) \text{ is true}}$$

so $P(m)$ is true.

$\therefore m \notin S$.

But this contradicts defn of m as least element in S. $\times$

Conclude $S = \phi$. $\blacksquare$