CSE 102
Introduction to Analysis of Algorithms
Some Common Functions (CLRS 3.2)

We present several common functions and estimates which occur frequently in the analysis of algorithms.

Floors and Ceilings

Given $x \in \boldsymbol{R}$, we denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the *floor of x* and the *ceiling of x*, respectively. These are defined to be the unique integers satisfying

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$$

Equivalently, if $x \in \boldsymbol{R}$ and $N \in \boldsymbol{Z}$ then

(1) $N = \lfloor x \rfloor$ if and only if $N \leq x < N + 1$, and
(2) $N = \lceil x \rceil$ if and only if $N - 1 < x \leq N$.

In other words:

(1) $\lfloor x \rfloor$ is the *greatest integer less than or equal to x*, and
(2) $\lceil x \rceil$ is the *least integer greater than or equal to x.*

<u>Lemma 1:</u> Let $x \in \boldsymbol{R}$ and $a, b \in \boldsymbol{Z}$. Then
(1) $a \leq x < b$ if and only if $a \leq \lfloor x \rfloor < b$, and
(2) $a < x \leq b$ if and only if $a < \lceil x \rceil \leq b$.

Proof of (1):
(i) $a \leq x$ implies $a \leq \lfloor x \rfloor$, since among all integers that are less than or equal to x, $\lfloor x \rfloor$ is the greatest.
(ii) $x < b$ implies $\lfloor x \rfloor < b$, since $\lfloor x \rfloor \leq x$.
(iii) $a \leq \lfloor x \rfloor$ implies $a \leq x$, since $\lfloor x \rfloor \leq x$.
(iv) $\lfloor x \rfloor < b$ implies $x < b$, since $b \leq x$ implies $b \leq \lfloor x \rfloor$, by (i). ■

Exercise: prove part (2).

<u>Lemma 2:</u> Let $x \in \boldsymbol{R}$ and $m \in \boldsymbol{Z}^+$. Then
(1) $\left\lfloor \frac{\lfloor x \rfloor}{m} \right\rfloor = \left\lfloor \frac{x}{m} \right\rfloor$, and
(2) $\left\lceil \frac{\lceil x \rceil}{m} \right\rceil = \left\lceil \frac{x}{m} \right\rceil$.

Proof of (1): Let $N = \big\lfloor \lfloor x \rfloor / m \big\rfloor$. Then

$$\begin{aligned}
& N \leq \frac{\lfloor x \rfloor}{m} < N + 1 \\
\Rightarrow\ & mN \leq \lfloor x \rfloor < m(N + 1) \\
\Rightarrow\ & mN \leq x < m(N + 1) \qquad \text{(by lemma 1)} \\
\Rightarrow\ & N \leq x/m < N + 1 \\
\Rightarrow\ & N = \lfloor x/m \rfloor,
\end{aligned}$$

and therefore $\lfloor \lfloor x \rfloor / m \rfloor = N = \lfloor x/m \rfloor$. ∎

Exercise: prove part (2).

<u>**Lemma 3:**</u> Let $a, b, n \in \boldsymbol{Z}^+$. Then

(1) $\left\lfloor \frac{\lfloor n/a \rfloor}{b} \right\rfloor = \left\lfloor \frac{n}{ab} \right\rfloor$, and

(2) $\left\lceil \frac{\lceil n/a \rceil}{b} \right\rceil = \left\lceil \frac{n}{ab} \right\rceil$.

Proof: Set $x = n/a$ and $m = b$ in lemma 2. ∎

Exercise

Let $n \in \boldsymbol{Z}$. Show that (a) $\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = n$, (b) $\left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n+1}{2} \right\rfloor$, and (c) $\left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n-1}{2} \right\rceil$.

Logarithms

Let $x, a, b \in \boldsymbol{R}$ where $x > 0$, $a > 1$, and $b > 1$. Then $\log_a(x)$ denotes the exponent on a which gives x. In other words, $\log_a(x)$ is the inverse function of a^x, which means $a^{\log_a(x)} = x$ and $\log_a(x) = x$. Thus

$$x = a^{\log_a(x)} = \left(b^{\log_b(a)}\right)^{\log_a(x)} = b^{\log_b(a) \cdot \log_a(x)}$$

Taking $\log_b()$ of both sides of this equation yields

$$(*) \qquad \log_b(x) = \log_b(a) \cdot \log_a(x)$$

which says in particular $\log_b(x) = \text{constant} \cdot \log_a(x)$, i.e. any two log functions differ by a constant multiple. It follows that $\log_b(n) = \Theta(\log_a(n))$, so speaking in terms of asymptotic growth rates, there is really only one log function. Equation (*) implies

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

which shows how to convert from one log function to another. In particular $\lg(x) = \frac{\ln(x)}{\ln(2)}$. Here we use the standard notation $\lg(\,) = \log_2(\,)$, and $ln(\,) = \log_e(\,)$, where $e = 2.71828..$. Equation (*) also implies $a^{\log_b(x)} = a^{\log_a(x) \cdot \log_b(a)} = \left(a^{\log_a(x)}\right)^{\log_b(a)} = x^{\log_b(a)}$, which gives us the useful formula

$$a^{\log_b(x)} = x^{\log_b(a)}.$$

Stirling's Formula

Let $n \in \boldsymbol{Z}^+$. Then $n\,! = \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)$.

Stirling's formula gives a simple way to determine asymptotic (upper, lower, and tight) bounds on functions involving $n\,!$. A slightly weaker version of the theorem is the asymptotic equivalence

$$n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$$

which is proved in a separate handout. Another proof can be found at

http://www.sosmath.com/calculus/sequence/stirling/stirling.html

The strong version (which is proved in *Concrete Mathematics* by Grahan, Knuth & Patashnik) is used in the applications below.

Corollary:
(1) $n\,! \;=\; o(n^n)$
(2) $n\,! \;=\; \omega(b^n)$ for any $b > 0$
(3) $\log(n!) = \;\Theta(n\log(n)\,)$

Proof of (1):

$$\frac{n\,!}{n^n} = \frac{\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)}{n^n} = \frac{\sqrt{2\pi n}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)}{e^n} \;\;\to\;\; 0 \;\text{ as }\; n\to\infty, \text{ showing that } n\,! \;=\; o(n^n).$$ ■

Proof of (3): Taking log (any base) of both sides of Stirling's formula, we get

$$\begin{aligned}\log(n!) \;&=\; \log\sqrt{2\pi n} + \log\left(\frac{n}{e}\right)^n + \log\left(1+\Theta\left(\frac{1}{n}\right)\right) \\ &= \tfrac{1}{2}\log(2\pi) + \tfrac{1}{2}\log(n) + n\log(n) - n\log(e) + \log\left(1+\Theta\left(\tfrac{1}{n}\right)\right).\end{aligned}$$

Therefore

$$\frac{\log(n!)}{n\log(n)} \;=\; 1 + (\text{stuff that } \;\to 0 \;\text{ as }\; n\to\infty),$$

hence $\lim_{n\to\infty}\left(\frac{\log(n!)}{n\log(n)}\right) = 1$, proving that $\log(n!) \;=\; \Theta(n\log(n))$. ■

Exercise: Prove part (2) of the corollary.

Exercise: Prove that $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$, where $\binom{m}{k}$ denotes the binomial coefficient $\binom{m}{k} = \frac{m\,!}{k\,!(m-k)\,!}$, for $0 \le k \le m$.

Exercise: Determine a number $a > 0$ such that $\binom{3n}{n} = \Theta\left(\frac{a^n}{\sqrt{n}}\right)$.

Exercise: For each $k \ge 2$, determine a number $a_k > 0$ such that $\binom{kn}{n} = \Theta\left(\frac{a_k^n}{\sqrt{n}}\right)$.