

CSE 102

Midterm 1 Review

Solutions to Selected Problems

1. Let $T(n)$ satisfy the recurrence $T(n) = aT(n/b) + f(n)$, where $a \geq 1$, $b > 1$ and $f(n)$ is a polynomial satisfying $\deg(f) > \log_b(a)$. Prove that case (3) of the Master Theorem applies, and in particular, prove that the regularity condition necessarily holds.

Proof:

Let $d = \deg(f)$ and replace $f(n)$ by the asymptotically equivalent function n^d . We compare the polynomials n^d and $n^{\log_b(a)}$. Let $\epsilon = d - \log_b(a)$, which is positive since $d > \log_b(a)$. Therefore $d = \log_b(a) + \epsilon$, and $n^d = \Omega(n^d) = \Omega(n^{\log_b(a)+\epsilon})$, verifying the first hypothesis of case (3).

Observe $d > \log_b(a) \Rightarrow b^d > a \Rightarrow a/b^d < 1$. Pick any c in the range $a/b^d \leq c < 1$. Then for any $n \geq 1$, we have $a(n/b)^d = (a/b^d)n^d \leq cn^d$, verifying the regularity condition. ■

2. Define the sequence $T(n)$ for $n \geq 1$ by the recurrence $T(n) = (n-1) + \frac{n-1}{n^2} \cdot \sum_{k=1}^{n-1} T(k)$. Use induction to prove $T(n) \leq 2n$ for all $n \geq 1$.

Proof:

I. Observe $T(1) = (1-1) + \frac{1-1}{1^2} \cdot (\text{empty sum}) = 0 \leq 2 = 2 \cdot 1$, establishing the base case.

II. Let $n > 1$, and assume for all k in the range $1 \leq k < n$ that $T(k) \leq 2k$. We must show that the inequality $T(n) \leq 2n$ is also true. We have

$$\begin{aligned} T(n) &= (n-1) + \frac{n-1}{n^2} \cdot \sum_{k=1}^{n-1} T(k) \\ &\leq (n-1) + \frac{n-1}{n^2} \cdot \sum_{k=1}^{n-1} 2k && \text{by the induction hypothesis} \\ &= (n-1) + \frac{n-1}{n^2} \cdot 2 \cdot \frac{n(n-1)}{2} \\ &= (n-1) + \frac{(n-1)^2}{n} \\ &= (n-1) \left(1 + \frac{n-1}{n} \right) \\ &= (n-1) \left(1 + 1 - \frac{1}{n} \right) \\ &= (n-1) \left(2 - \frac{1}{n} \right) \\ &= 2n - 2 - 1 + \frac{1}{n} \\ &= 2n - 3 + \frac{1}{n} \leq 2n && (\text{since } n > 1 \Rightarrow \frac{1}{n} \leq 1 \Rightarrow -3 + \frac{1}{n} \leq 0) \end{aligned}$$

as required. It follows that $T(n) \leq 2n$ for all $n \geq 1$. ■

3. Define $T(n)$ by the recurrence

$$T(n) = \begin{cases} 0 & n = 1 \\ 2T(\lfloor n/2 \rfloor) + n \lg(n) & n \geq 2 \end{cases}$$

Here $\lg$ means $\log_2$.

a. Show that the Master Theorem cannot be applied to this recurrence.

Proof:

We first simplify the recurrence to $T(n) = 2T(n/2) + n \lg(n)$. Comparing $n \lg(n)$ to $n^{\log_2(2)} = n$, we see that $\frac{n \lg(n)}{n} = \lg(n) \rightarrow \infty$ as $n \rightarrow \infty$, so that $n \lg(n) = \omega(n)$. Since $n \lg(n)$ is the winner, the only possible case of the Master Theorem that could apply is case 3. But although $n \lg(n)$ is the winner, it does not win by a polynomial factor. Indeed, let $\epsilon > 0$ be chosen arbitrarily. Then

$$\frac{n \lg(n)}{n^{\log_2(2)+\epsilon}} = \frac{n \lg(n)}{n^{1+\epsilon}} = \frac{\lg(n)}{n^\epsilon} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which yields $n \lg(n) = o(n^{\log_2(2)+\epsilon})$. Exercise 6 on page 5 of the handout on asymptotic growth rates implies $o(n^{\log_2(2)+\epsilon}) \cap \Omega(n^{\log_2(2)+\epsilon}) = \emptyset$, and hence $n \lg(n) \notin \Omega(n^{\log_2(2)+\epsilon})$. Since $\epsilon > 0$ was arbitrary, this holds for *all* such ϵ . Therefore we are not in case 3, and the Master Theorem cannot be applied. ■

b. Use the Substitution method to prove that $T(n) = O(n (\lg n)^2)$.

Proof:

We show by induction that $T(n) \leq n(\lg n)^2$ for all $n \geq 1$, from which $T(n) = O(n (\lg n)^2)$ follows.

I. For $n = 1$ we have $T(1) = 0 \leq 0 = 1 \cdot (\lg 1)^2$, and the base case is satisfied.

II. Let $n > 1$ be arbitrary, and assume $T(k) \leq k(\lg k)^2$ for any k in the range $1 \leq k < n$. We must show that $T(n) \leq n(\lg n)^2$.

$$\begin{aligned} T(n) &= 2T(\lfloor n/2 \rfloor) + n \lg(n) \\ &\leq 2 \cdot \lfloor n/2 \rfloor (\lg \lfloor n/2 \rfloor)^2 + n \lg(n) && \text{By the induction hypothesis with } k = \lfloor n/2 \rfloor. \\ &\leq 2 \cdot (n/2) (\lg(n/2))^2 + n \lg(n) && \text{Since } \lfloor x \rfloor \leq x \text{ for any } x \in \mathbb{R} \\ &= n(\lg(n) - 1)^2 + n \lg(n) \\ &= n((\lg(n))^2 - 2 \lg(n) + 1) + n \lg(n) \\ &= n(\lg n)^2 - 2n \lg(n) + n + n \lg(n) \\ &= n(\lg n)^2 - n \lg(n) + n \\ &\leq n(\lg n)^2 \end{aligned}$$

The last inequality follows from

$$n > 1 \Rightarrow n \geq 2 \Rightarrow \lg(n) \geq 1 \Rightarrow n \lg(n) \geq n \Rightarrow -n \lg(n) + n \leq 0.$$

Therefore $T(n) \leq n(\lg n)^2$ for all $n \geq 1$ by the 2nd PMI. ■