

CSE 102
Homework Assignment 2
Solutions

1. Let $f(n)$ be a positive, increasing function that satisfies $f(n/2) = \Theta(f(n))$. Prove that $\sum_{i=1}^n f(i) = \Theta(nf(n))$. Hint: emulate the example on page 4 of the handout on asymptotic growth rates in which it is shown that $\sum_{i=1}^n i^k = \Theta(n^{k+1})$.

Proof:

Since $f(n)$ is increasing, we have $\sum_{i=1}^n f(i) \leq \sum_{i=1}^n f(n) = nf(n) = O(nf(n))$. Note also that

$$\begin{aligned}
 \sum_{i=1}^n f(i) &\geq \sum_{i=\lceil n/2 \rceil}^n f(i) && \text{by discarding some positive terms} \\
 &\geq \sum_{i=\lceil n/2 \rceil}^n f(\lceil n/2 \rceil) && \text{since } f(n) \text{ is increasing} \\
 &= (n - \lceil n/2 \rceil + 1)f(\lceil n/2 \rceil) \\
 &= (\lfloor n/2 \rfloor + 1)f(\lceil n/2 \rceil) \\
 &> ((n/2) - 1 + 1)f(n/2) && \text{since } \lfloor x \rfloor > x - 1, \lfloor x \rfloor \geq x \text{ and } f(n) \text{ is increasing} \\
 &= (1/2)n\Omega(f(n)) && \text{since } f(n/2) = \Theta(f(n)) \subseteq \Omega(f(n)) \\
 &= \Omega(nf(n))
 \end{aligned}$$

Thus $\sum_{i=1}^n f(i)$ is bounded above and below by functions in the classes $O(nf(n))$ and $\Omega(nf(n))$, respectively. By an exercise in the handout on asymptotic growth rates, we have $\sum_{i=1}^n f(i) = \Theta(nf(n))$ as required. ■

2. Use the result of the preceding problem to prove that $\log(n!) = \Theta(n \log(n))$, without using Stirling's formula.

Proof:

Observe that the function $\log n$ is positive (for $n > 1$), increasing and $\log(n/2) = \log n - \log 2 = \Theta(\log n)$. Therefore we may take $f(n) = \log(n)$ in the preceding problem, giving

$$\begin{aligned}
 \log(n!) &= \log(1 \cdot 2 \cdot 3 \cdots n) \\
 &= \sum_{i=1}^n \log(i) \\
 &= \Theta(n \log n).
 \end{aligned}$$

■

3. Use Stirling's formula to determine a constant $a > 0$ such that $\binom{3n}{n} = \Theta\left(\frac{a^n}{\sqrt{n}}\right)$.

Proof:

By Stirling's formula we have

$$\begin{aligned}
 \binom{3n}{n} &= \frac{(3n)!}{n! \cdot (2n)!} \\
 &= \frac{\sqrt{2\pi \cdot 3n} \cdot \left(\frac{3n}{e}\right)^{3n} \cdot \left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\left(\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)\right) \cdot \left(\sqrt{2\pi \cdot 2n} \cdot \left(\frac{2n}{e}\right)^{2n} \cdot \left(1 + \Theta\left(\frac{1}{2n}\right)\right)\right)}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{\sqrt{2\pi} \cdot \sqrt{3n} \cdot \frac{3^{3n} n^{3n}}{e^{3n}} \cdot \left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\sqrt{2\pi} \cdot \sqrt{n} \cdot \frac{n^n}{e^n} \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right) \cdot \sqrt{2\pi} \cdot \sqrt{2n} \cdot \frac{2^{2n} n^{2n}}{e^{2n}} \cdot \left(1 + \Theta\left(\frac{1}{2n}\right)\right)} \\
&= \frac{\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{27^n / 4^n}{\sqrt{n}} \cdot \frac{\left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\left(1 + \Theta\left(\frac{1}{n}\right)\right) \cdot \left(1 + \Theta\left(\frac{1}{2n}\right)\right)}.
\end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{\binom{3n}{n}}{\frac{(27/4)^n}{\sqrt{n}}} = \frac{\sqrt{3}}{2\sqrt{\pi}} \cdot \lim_{n \rightarrow \infty} \frac{\left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\left(1 + \Theta\left(\frac{1}{n}\right)\right) \cdot \left(1 + \Theta\left(\frac{1}{2n}\right)\right)} = \frac{\sqrt{3}}{2\sqrt{\pi}},$$

and since $0 < \frac{\sqrt{3}}{2\sqrt{\pi}} < \infty$, it follows that $\binom{3n}{n} = \Theta\left(\frac{(27/4)^n}{\sqrt{n}}\right)$. Hence $a = 27/4$. ■

4. Define $S(n)$ for $n \in \mathbb{Z}^+$ by the recurrence

$$S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lfloor n/2 \rfloor) + 1 & \text{if } n \geq 2 \end{cases}$$

Prove that $S(n) \geq \lg(n)$ for all $n \geq 1$, and hence $S(n) = \Omega(\lg(n))$.

Proof:

- I. When $n = 1$ we have $S(1) \geq \lg(1)$, which reduces to $0 \geq 0$, which is true.
- II. Let $n > 1$ and assume for all k in the range $1 \leq k < n$ that $S(k) \geq \lg(k)$. We must show that $S(n) \geq \lg(n)$. We have

$$\begin{aligned}
S(n) &= S(\lfloor n/2 \rfloor) + 1 \\
&\geq \lg(\lfloor n/2 \rfloor) + 1 && \text{by the induction hypothesis with } k = \lfloor n/2 \rfloor \\
&\geq \lg(n/2) + 1 && \text{since } \lfloor x \rfloor \geq x \text{ for any } x \\
&= \lg(n) - \lg(2) + 1 \\
&= \lg(n).
\end{aligned}$$

By the Second Principle of Mathematical Induction, $S(n) \geq \lg(n)$ for all $n \geq 1$. ■

5. Let $T(n)$ be defined by the recurrence

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & \text{if } n \geq 2 \end{cases}$$

Show that $\forall n \geq 1: T(n) \leq (4/3)n^2$, and hence $T(n) = O(n^2)$.

Proof:

- I. For $n = 1$ we have $T(1) = 1 \leq 4/3 = (4/3) \cdot 1^2$, so the base case is satisfied.
- II. Let $n > 1$ and assume for all k in the range $1 \leq k < n$ that $T(k) \leq (4/3)k^2$. In particular we have $T(\lfloor n/2 \rfloor) \leq (4/3)\lfloor n/2 \rfloor^2$. We must show that $T(n) \leq (4/3)n^2$. We have

$$\begin{aligned}
T(n) &= T(\lfloor n/2 \rfloor) + n^2 \\
&\leq (4/3)\lfloor n/2 \rfloor^2 + n^2 && \text{by the induction hypothesis} \\
&\leq (4/3)(n/2)^2 + n^2 && \text{since } \lfloor x \rfloor \leq x \text{ for any } x \\
&= (1/3)n^2 + n^2 \\
&= (4/3)n^2.
\end{aligned}$$

By the Second Principle of Mathematical Induction, $T(n) \leq (4/3)n^2$ for all $n \geq 1$. ■

6. Prove that the First Principle of Mathematical Induction implies the Second Principle of Mathematical Induction. (This is Exercise 4 at the end of the handout on Induction Proofs.)

Proof:

The 1st PMI asserts that for any propositional function $P(n)$, the following sentence holds.

$$P(1) \wedge [\forall n > 1: P(n-1) \rightarrow P(n)] \rightarrow \forall n \geq 1: P(n)$$

The 2nd PMI says the following sentence is true for any propositional function $Q(n)$.

$$Q(1) \wedge [\forall n > 1: (Q(1) \wedge Q(2) \wedge \cdots \wedge Q(n-1)) \rightarrow Q(n)] \rightarrow \forall n \geq 1: Q(n)$$

We assume the 1st PMI, and show the 2nd PMI as a consequence. To that end, let $Q(n)$ be any propositional function, and define $P(n)$ by

$$P(n) = Q(1) \wedge Q(2) \wedge \cdots \wedge Q(n).$$

In particular, we have

$$P(1) = Q(1),$$

$$P(n-1) = Q(1) \wedge Q(2) \wedge \cdots \wedge Q(n-1),$$

and

$$P(n) = P(n-1) \wedge Q(n).$$

To prove the 2nd PMI, we assume both $Q(1)$ and $\forall n > 1: (Q(1) \wedge Q(2) \wedge \cdots \wedge Q(n-1)) \rightarrow Q(n)$ are true. We must show that $\forall n \geq 1: Q(n)$ holds.

These assumptions give us that both $P(1)$ and $\forall n > 1: P(n-1) \rightarrow Q(n)$ are true. It is an elementary fact of logic that $\forall n > 1: P(n-1) \rightarrow P(n-1)$, therefore $\forall n > 1: P(n-1) \rightarrow P(n-1) \wedge Q(n)$ holds, which is equivalent to $\forall n > 1: P(n-1) \rightarrow P(n)$. We have shown that under our assumptions both $P(1)$ and $\forall n > 1: P(n-1) \rightarrow P(n)$ are true. The 1st PMI now yields $\forall n \geq 1: P(n)$. Obviously $P(n) \rightarrow Q(n)$, so $\forall n \geq 1: Q(n)$ is also true.

We have shown that if $Q(1)$ and $\forall n > 1: (Q(1) \wedge Q(2) \wedge \cdots \wedge Q(n-1)) \rightarrow Q(n)$ are both true, then $\forall n \geq 1: Q(n)$ must also be true, establishing the 2nd PMI. ■