cse 102 9-30-21

Recall that: $o(g(n)) \subseteq O(g(n))$

<u>Exercise</u>: show that

$$o(g(n)) \cap \Omega(g(n)) = \phi$$

<u>consequence</u>: $o(g(n)) \cap \Theta(g(n)) = \phi$

<u>Defn</u>

$\omega(g(n)) = \{f(n) \mid \forall c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c\,g(n) < f(n)\}$

we say: $g(n)$ is a strict <u>asymptotic lower bound</u> for $f(n)$.

<u>Recall</u>:

$\Omega(g(n)) = \{f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c\,g(n) \leq f(n)\}$

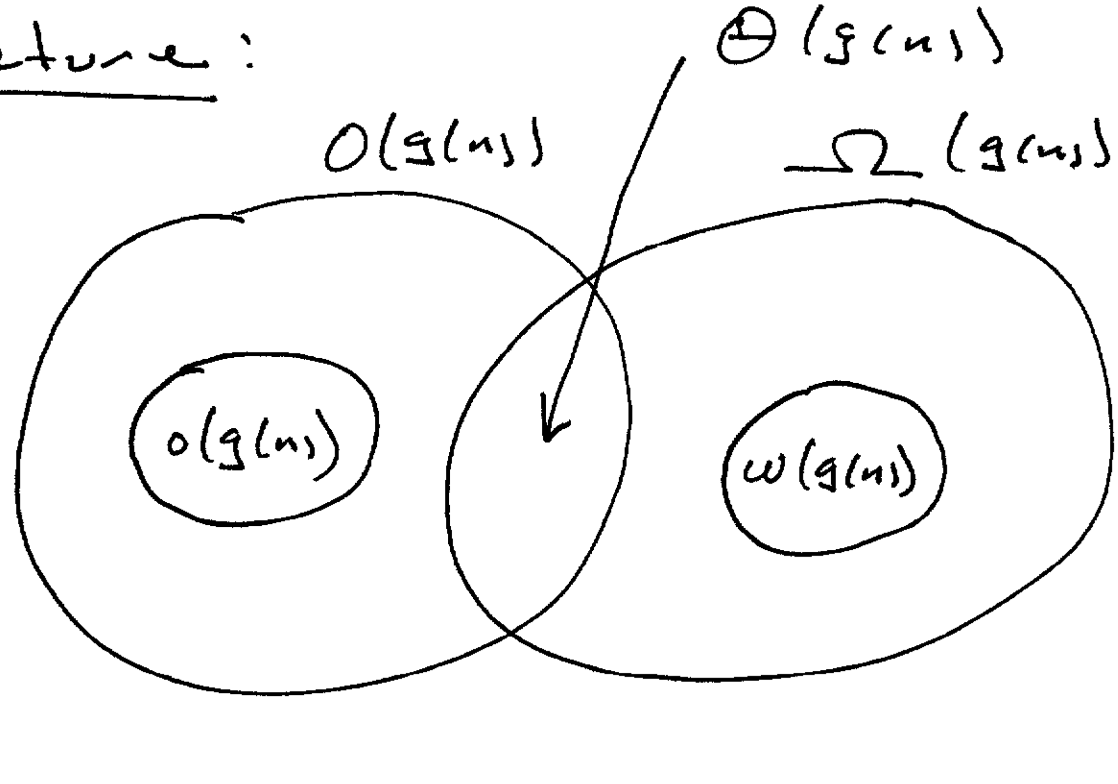
Observe: $\omega(g(n)) \subseteq \Omega(g(n))$

<u>Lemma</u>: $f(n) = \omega(g(n))$ iff $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = \infty$

<u>Exercise</u>

- Prove lemma.
- Prove $\omega(g(n)) \cap O(g(n)) = \phi$

<u>Picture</u>:



Theorem

If $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = L$, where $0 \le L < \infty$,

then $f(n) = O(g(n))$.

(note: Converse is false.

Proof.

The defn. of the limit is:

$\forall \varepsilon > 0, \exists n_0 > 0, \forall n \ge n_0 : \left| \frac{f(n)}{g(n)} - L \right| < \varepsilon$

Since this holds for all $\varepsilon > 0$, we can pick $\varepsilon = 1$. Then there exists $n_0 > 0$ such that for all $n \ge n_0$:

$$\left|\frac{f(n)}{g(n)} - L\right| < 1$$

$$\therefore \quad -1 < \frac{f(n)}{g(n)} - L < 1$$

ignore

$$\therefore \quad \frac{f(n)}{g(n)} < L+1$$

$$\therefore \quad f(n) < (L+1)\, g(n)$$

Hence, letting $c = (L+1)$, we have

$\exists c > 0, \exists n_0 > 0$ s.t. $\forall n \geq n_0$:

$$0 \leq f(n) \leq c \cdot g(n) \quad .$$

since $f(n)$
is a.n.n.

Therefore $f(n) = O(g(n))$.

■

Theorem:

If $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = L$, where $0 < L \le \infty$,

then $f(n) = \Omega(g(n))$. (note: again converse is false)

Proof:

The limit statement implies that $\lim_{n\to\infty}\left(\frac{g(n)}{f(n)}\right) = L'$, where $L' = \frac{1}{L}$.

Therefore $0 \le L' < \infty$. By the Prev. thm. we have

$$g(n) = O(f(n)).$$

By an earlier theorem

$$f(n) = \Omega(g(n)).$$

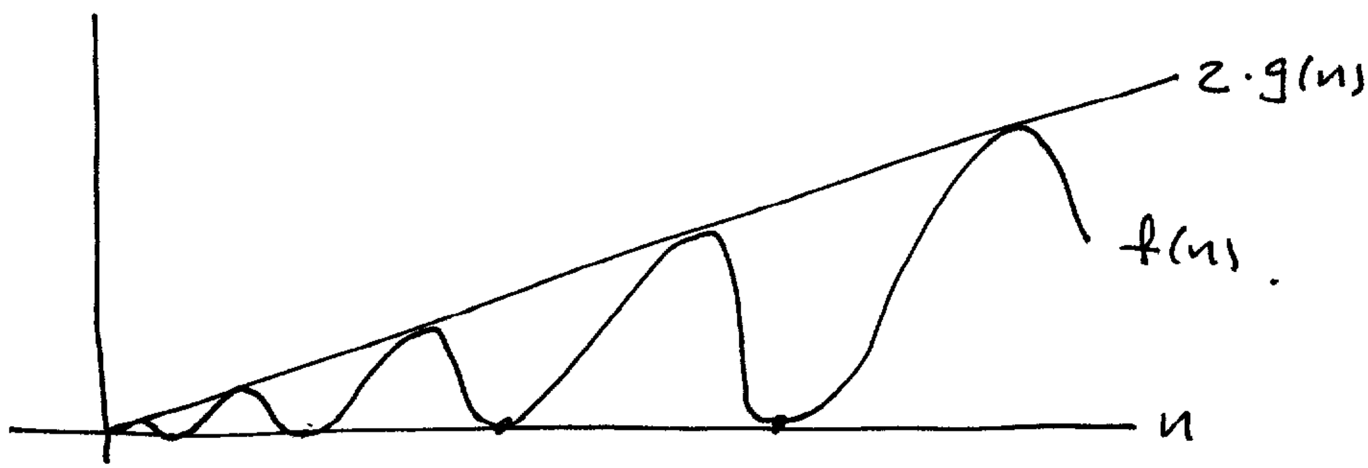
$\blacksquare$

<u>Exercise</u>: Prove that

if $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = L$, where $0 < L < \infty$,

then $f(n) = \Theta(g(n))$. $\left(\begin{array}{l}\underline{\text{note:}} \text{ the} \\ \text{converse is false}\end{array}\right)$

<u>Ex.</u> A let $g(n) = n$, $f(n) = (1+\sin(n))\cdot n$

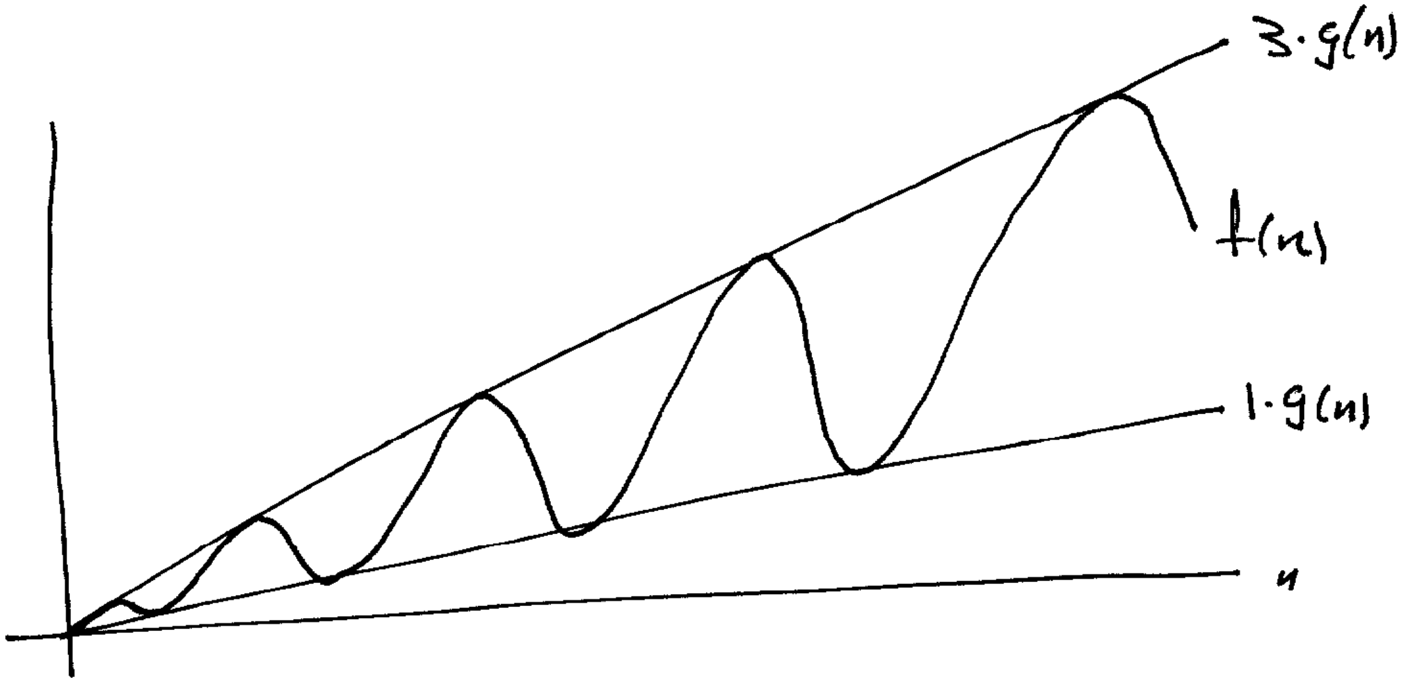


obviously $f(n) = O(g(n))$. But

$$\frac{f(n)}{g(n)} = 1+\sin(n)$$

has no limit as $n\to\infty$. Hence $f(n) \neq o(g(n))$. Also $f(n) \neq \Omega(g(n))$

Ex B let $g(n) = n$, $f(n) = (2 + \sin(n)) \cdot n$



obviously $f(n) = \Theta(g(n))$, but

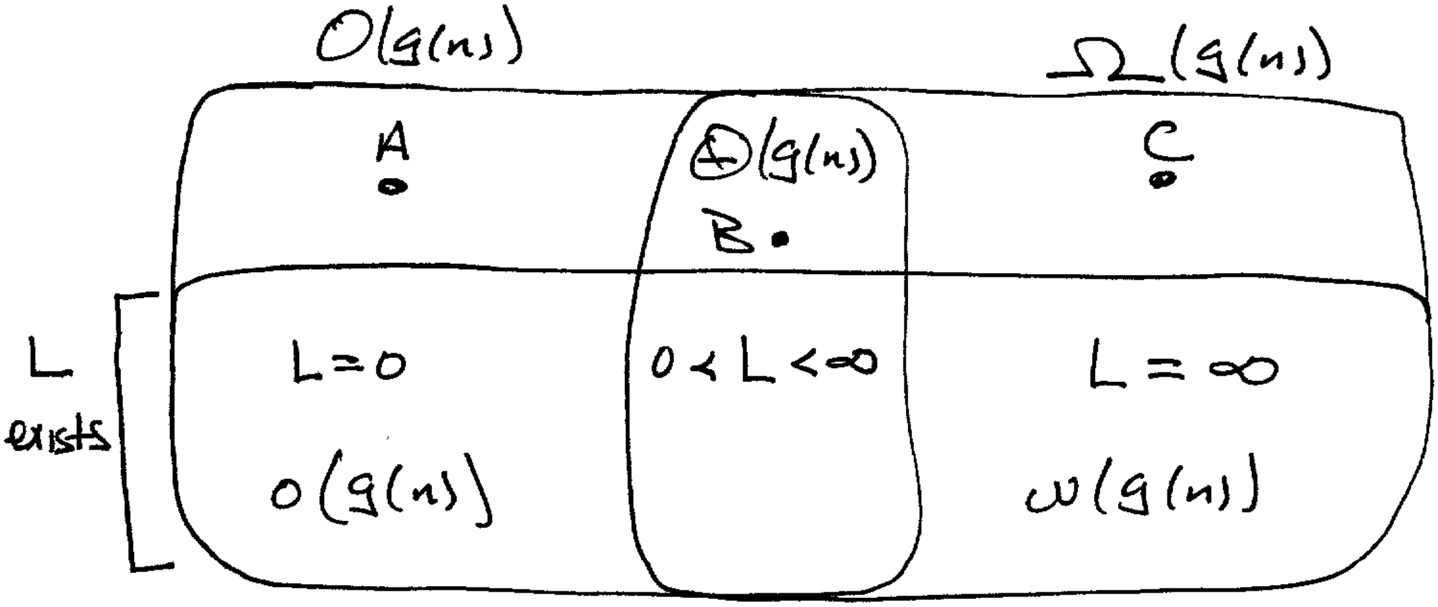
$$\frac{f(n)}{g(n)} = 2 + \sin(n),$$

whose limit as $n \to \infty$ does not exist.

Exercise find example C: functions $f(n), g(n)$ such that $f(n) = \Omega(g(n))$, but

$\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right)$ D.N.E.

Refined Picture:



let $L=\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right)$ if it exist.

Recall exercise: $(n+a)^b=\Theta(n^b)$

for any a, and $b>0$.

Proof.

$$\lim_{n\to\infty}\frac{(n+a)^b}{n^b}=\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^b$$

$$=\left(\lim_{n\to\infty}\left(1+\frac{a}{n}\right)\right)^b=1^b=1$$

$\therefore\ (n+a)^b=\Theta(n^b)$.

Exercise 10c (handout)

If $P(n)$ is a Polynomial of degree $k \geq 0$, then $P(n) = \Theta(n^k)$.

Proof:

$$P(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_2 n^2 + a_1 n + a_0$$

$(a_k > 0)$

$$\therefore \frac{P(n)}{n^k} = a_k + \underbrace{\frac{a_{k-1}}{n}}_{\downarrow\, 0} + \cdots + \underbrace{\frac{a_2}{n^{k-2}}}_{\downarrow\, 0} + \underbrace{\frac{a_1}{n^{k-1}}}_{\downarrow\, 0} + \underbrace{\frac{a_0}{n^k}}_{\downarrow\, 0}$$

$$\therefore \lim_{n\to\infty} \left(\frac{P(n)}{n^k}\right) = a_k \quad \text{and} \quad 0 < a_k < \infty$$

since $P(n)$ is a.n.n.

$$\therefore \quad P(n) = \Theta(n^k).$$

$\blacksquare$

Exercise 10d.

Let $\alpha, \beta \in \mathbb{R}^+$. Then

$$n^\alpha = \begin{cases} o(n^\beta) & \text{if } \alpha < \beta \\ \Theta(n^\beta) & \text{if } \alpha = \beta \\ \omega(n^\beta) & \text{if } \alpha > \beta \end{cases}$$

Proof:

$$\frac{n^\alpha}{n^\beta} = n^{\alpha-\beta} \xrightarrow[n\to\infty]{\text{as}} \begin{cases} 0 & \text{if } \alpha < \beta \\ 1 & \text{if } \alpha = \beta \\ \infty & \text{if } \alpha > \beta \end{cases}$$

$\blacksquare$

Exercise 10 e

for any Positive a, b:

$$a^n = \begin{cases} o(b^n) & \text{if} \quad a < b \\ \Theta(b^n) & \text{if} \quad a = b \\ \omega(b^n) & \text{if} \quad a > b \end{cases}$$

Proof:

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n \xrightarrow[\text{as } n\to\infty]{} \begin{cases} 0 & \text{if} \quad a < b \\ 1 & \text{if} \quad a = b \\ \infty & \text{if} \quad a > b \end{cases}$$

$\blacksquare$

Exercise 10 g

$$f(n) + o(f(n)) = \Theta(f(n))$$

Proof:

Let $h(n) = o(f(n))$. Then

$$\frac{h(n)}{f(n)} \longrightarrow 0 \text{ as } n \to \infty.$$

so

$$\frac{f(n) + h(n)}{f(n)} = 1 + \frac{h(n)}{f(n)} \longrightarrow 1 \text{ as } n \to \infty$$

since $0 < 1 < \infty$, we have

$$f(n) + h(n) = \Theta(f(n)).$$

$$\therefore \quad f(n) + o(f(n)) = \Theta(f(n)).$$

$\blacksquare$