CSE 102 9-28-21

Defn

let $g(n)$ be a function. let

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n)).$$

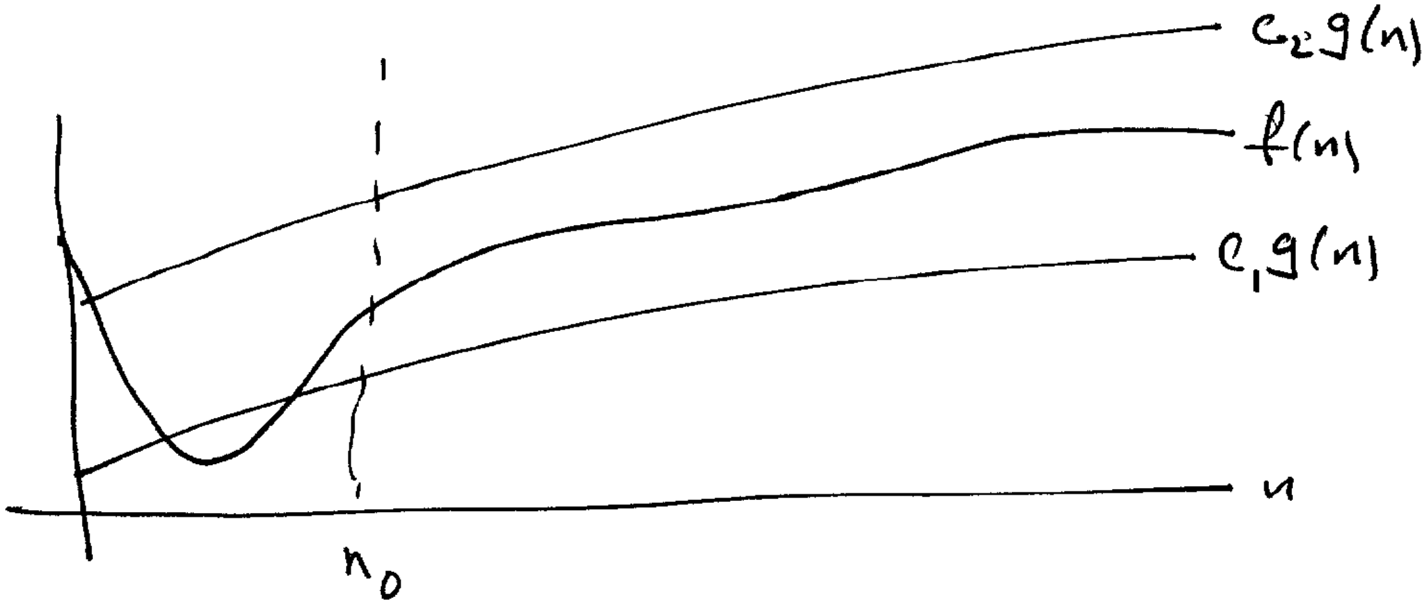
Equivalently:

$f(n) = \Theta(g(n))$ iff $\exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0$ such that $\forall n \geq n_0$:

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 \cdot g(n).$$

We say: $g(n)$ is a <u>tight asymptotic bound</u> for $f(n)$.

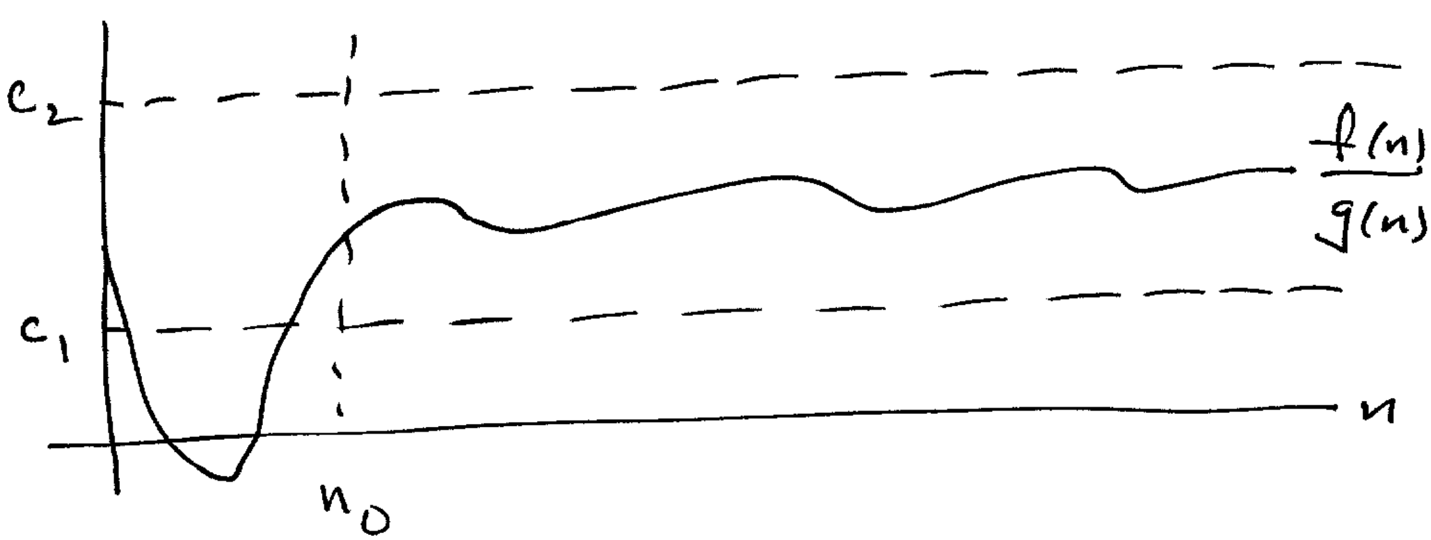
Picture:



If $g(n)$ is a.p., this is equiv. to

$$0 \le c_1 \le \frac{f(n)}{g(n)} \le c_2$$

Picture:



Exercise

Prove that $f(n) = \Theta(g(n))$ iff $g(n) = \Theta(f(n))$.

Exercise

let $g(n)$ be a fcn., and let $c > 0$.

Prove that

(1) $c \cdot g(n) = O(g(n))$

(2) $c \cdot g(n) = \Omega(g(n))$

(3) $c \cdot g(n) = \Theta(g(n))$

Ex. show that $\sqrt{n+10} = \Theta(\sqrt{n})$.

must find Pos. numbers c_1, c_2, n_0 s.t. for all $n \geq n_0$:

$$0 \leq c_1\sqrt{n} \leq \sqrt{n+10} \leq c_2\sqrt{n}$$

Let $c_1 = 1$, $c_2 = \sqrt{2}$, $n_0 = 10$.

Then if $n \geq n_0$, we have

$$-10 \leq 0 \quad \text{and} \quad 10 \leq n$$

$$\therefore \quad -10 \leq (1-1)n \quad \text{and} \quad 10 \leq (2-1)n$$

$$\therefore \quad -10 \leq (1-c_1^2)n \quad \text{and} \quad 10 \leq (c_2^2-1)n$$

$$\therefore \quad c_1^2 n \leq n+10 \quad \text{and} \quad n+10 \leq c_2^2 n$$

$$\therefore \quad 0 \leq c_1^2 n \leq n+10 \leq c_2^2 n$$

$$\therefore \quad 0 \leq c_1\sqrt{n} \leq \sqrt{n+10} \leq c_2\sqrt{n},$$

as required. ∎

note: check that $c_1 = \sqrt{\frac{1}{2}}$, $c_2 = \sqrt{\frac{3}{2}}$ and $n_0 = 20$ also work.

Theorem

If $h(n) = O(g(n))$ and if $f(n) \le h(n)$ for all suff. large n, then

$$f(n) = O(g(n)).$$

Proof.

we have

(1) $\exists$ pos c_1, n_1 s.t. $\forall n \ge n_1$

$$0 \le h(n) \le c_1 g(n)$$

and

(2) $\exists$ pos n_2 s.t. $\forall n \ge n_2$

$$0 \le f(n) \le h(n)$$

f is a.n.n.

we must show

(3) $\exists$ pos. c, n_0 st. $\forall n \geq n_0$

$$0 \leq f(n) \leq c \cdot g(n) .$$

Let $n_0 = \max(n_1, n_2)$, and $c = c_1$.

Then if $n \geq n_0$ we have $n \geq n_1$ (so (1) is true) and $n \geq n_2$ (so (2) is true), therefore

$$0 \leq f(n) \leq h(n) \leq c_1 g(n) = c \cdot g(n)$$

$\therefore$ $$0 \leq f(n) \leq c \cdot g(n),$$

showing that (3) is true. $\blacksquare$

Exercise:

- show that if $h(n) = \Omega(g(n))$ and if $f(n) \geq h(n)$ for all suff. large n, then $f(n) = \Omega(g(n))$.

- show that if $h_1(n) = \Omega(g(n))$, $h_2(n) = O(g(n))$, and if $h_1(n) \leq f(n) \leq h_2(n)$ for all suff. large n, then

$$f(n) = \Theta(g(n)).$$

Ex. let $k \geq 0$ be fixed. Prove that

$$\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$$

Proof

Observe

$$\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1} = O(n^{k+1}).$$

also

$$\sum_{i=1}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k$$

$$\geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \lceil \tfrac{n}{2} \rceil^k$$

$$= \left(n - \lceil \tfrac{n}{2} \rceil + 1\right) \lceil \tfrac{n}{2} \rceil^k$$

$$= \left(\lfloor \tfrac{n}{2} \rfloor + 1\right) \cdot \lceil \tfrac{n}{2} \rceil^k \quad \left\{ \begin{array}{l} \text{since} \\ n = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil \end{array} \right.$$

$$> \left(\frac{n}{2} - \cancel{1} + \cancel{1}\right)\left(\frac{n}{2}\right)^k$$

Since
$\lfloor x \rfloor > x - 1$
$\lceil x \rceil \geq x$

$$= \frac{n}{2} \cdot \left(\frac{n}{2}\right)^k$$

$$= \left(\frac{1}{2}\right)^{k+1} \cdot n^{k+1} = \Omega\left(n^{k+1}\right)$$

By the prev. exercise

$$\sum_{i=1}^{n} i^k = \Theta\left(n^{k+1}\right).$$

∎

<u>Remark</u>: recall that

$$k=1: \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \Theta(n^2)$$

$$k=2: \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \Theta(n^3).$$

$$k = 3 : \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \Theta(n^4)$$

$$\vdots$$

Defn

Let $g(n)$ be a function.

$$o(g(n)) = \{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) < c \cdot g(n) \}$$

we say: $g(n)$ is a strict asymotptotic upper bound to $f(n)$.

Recall:

$$O(g(n)) = \{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n) \}$$

observe: $o(g(n)) \subseteq O(g(n))$.

Lemma

$$f(n) = o(g(n)) \text{ iff } \lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = 0$$

Proof.

observe $f(n) = o(g(n))$ iff

$$\forall c>0, \exists n_0>0, \forall n \geq n_0 \quad 0 \leq \frac{f(n)}{g(n)} < c .$$

This is the very definition of

$$\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = 0 .$$

$\blacksquare$

Ex. $\ln(n) = o(n)$ since

$$\lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right) = \lim_{n\to\infty}\left(\frac{1/n}{1}\right) = 0 .$$

l'Hopital's rule.

<u>Exercise</u> show for any $k_1, k_2, b > 0$ that

$$(\log_b(n))^{k_1} = o(n^{k_2})$$

<u>Ex.</u> let $k > 0$. $n^k = o(e^n)$ since

$$\lim_{n\to\infty} \left(\frac{n^k}{e^n}\right) = \lim_{n\to\infty}\left(\frac{k n^{k-1}}{e^n}\right)$$

$= \cdots\cdots\cdots$ $\left\{\begin{array}{l}\lceil k \rceil \text{ applications} \\ \text{of l'HoP.}\end{array}\right.$

$= 0$

<u>Exercise</u> show $n^k = o(b^n)$ for any $k > 0$, $b > 1$.

analogy :

$f(n) = O(g(n)) \sim a \le b$

$f(n) = \Omega(g(n)) \sim a \ge b$

$f(n) = \Theta(g(n)) \sim a = b$

$f(n) = o(g(n)) \sim a < b$

$f(n) = \omega(g(n)) \sim a > b$