Asymptotic growth of functions:

<u>Defn</u>
let $g(n)$ be a function.

$$O(g(n)) = \{ f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n) \}$$

i.e. $f(n) \in O(g(n))$ <u>iff</u> there exist positive c, n_0 s.t. for all $n \geq n_0$:

$$0 \leq f(n) \leq c \cdot g(n)$$

holds. <u>note</u>: if $g(n) \neq 0$, this is equiv. to:

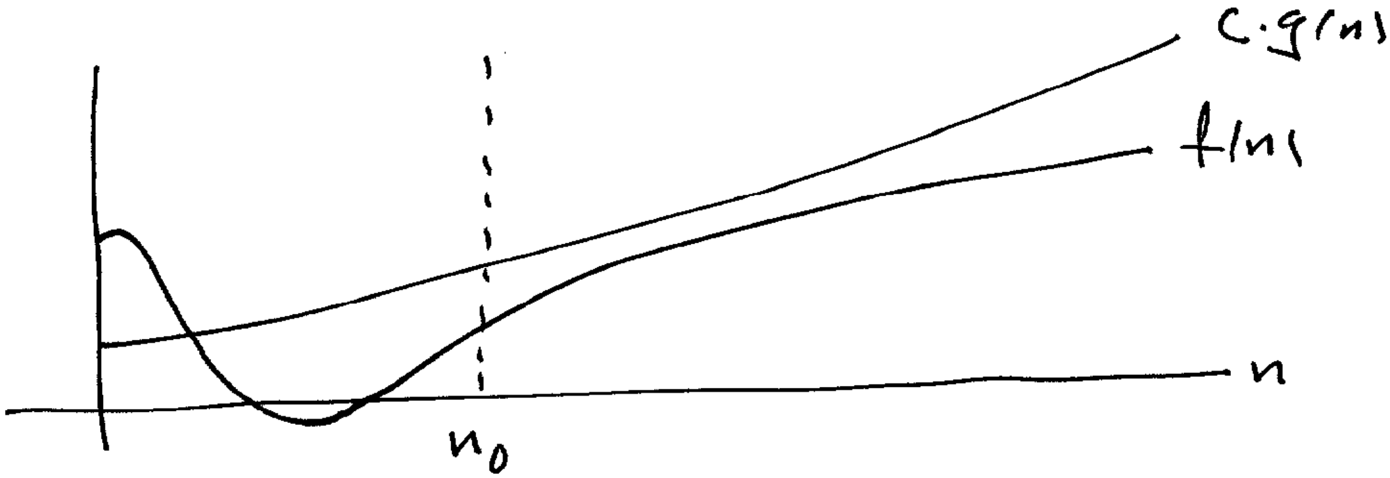
$$0 \leq \frac{f(n)}{g(n)} \leq c.$$

we say: $g(n)$ is an <u>asymptotic upper bound</u> for $f(n)$.

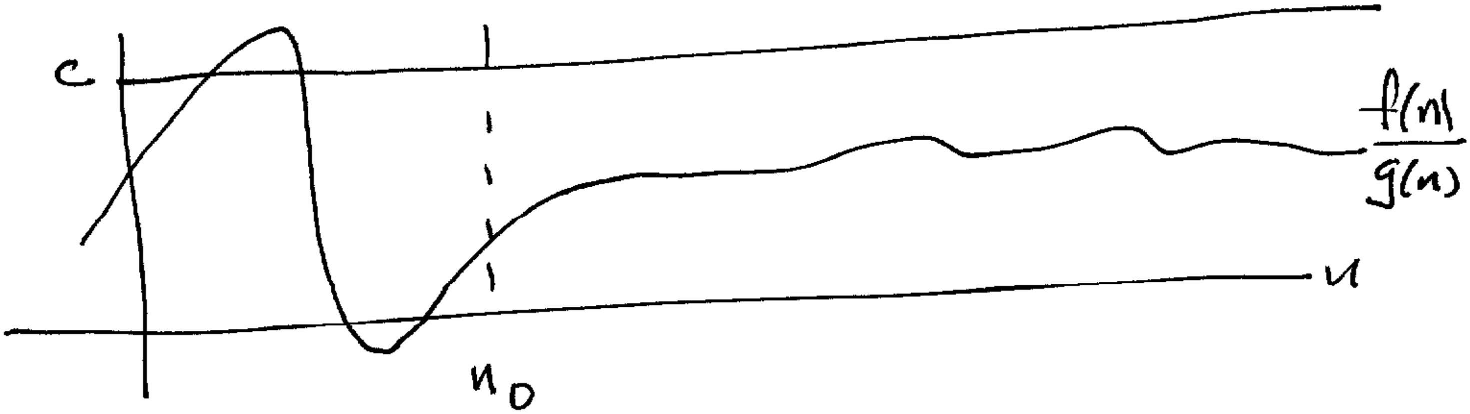
or: $f(n)$ is <u>asymptotically bounded above</u> by $g(n)$.

<u>notation</u>: $f(n) = O(g(n))$

<u>Picture</u>:



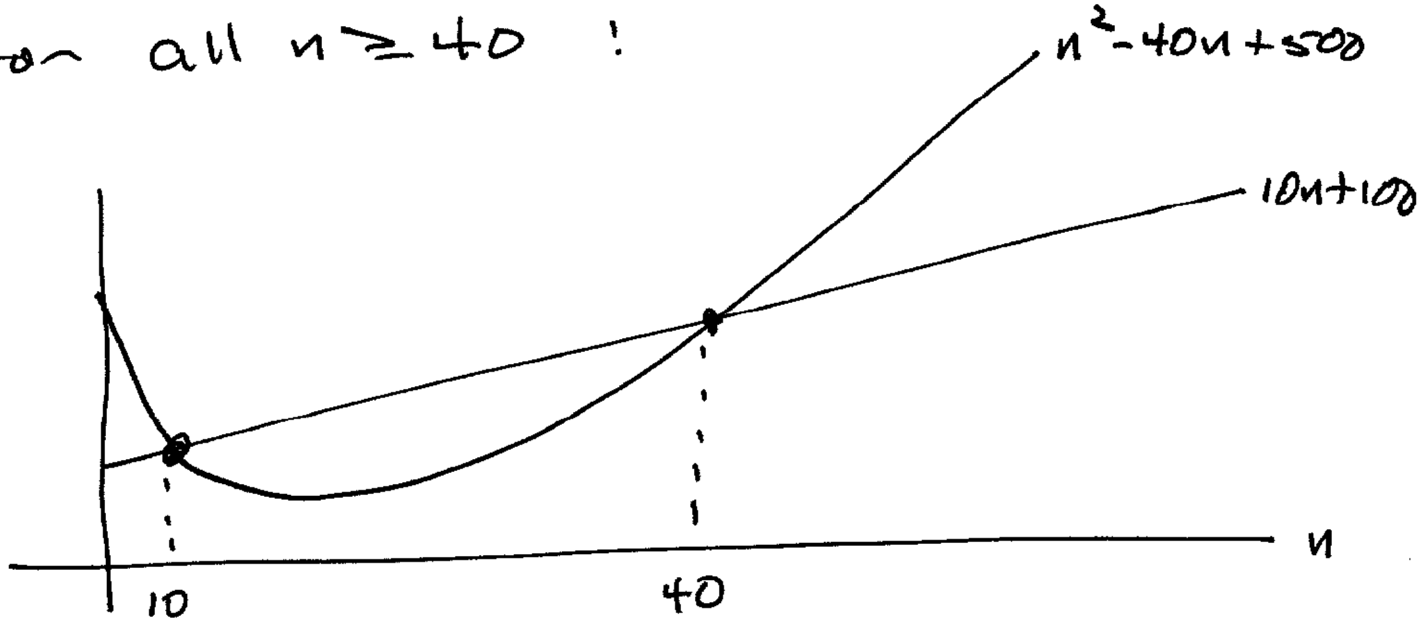
<u>also</u>:



Ex: $\overbrace{10n+100}^{f(n)} = O(\overbrace{n^2-40n+500}^{g(n)})$

Observe: $0 \le 10n+100 \le n^2-40n+500$

for all $n \ge 40$!



so $c=1$ and $n_0=40$.

In fact: $\boxed{an+b = O(cn^2+dn+e)}$

for any constants a, b, c, d, e with $a>0$ and $c>0$.

In fact: $\boxed{P(n) = O(Q(n))}$

for any Polynomials $P(n)$, $Q(n)$ with

$$\deg P(n) \le \deg Q(n)$$

Defn

- we say f(n) is asymptotically non-negative iff $\exists n_0 > 0$ s.t. for all $n \geq n_0$

$$0 \leq f(n)$$

holds. (a.n.n.)

- we say f(n) is asymptotically positive iff $\exists n_0 > 0$ s.t. $\forall n \geq n_0$

$$0 < f(n) \qquad (a.p.)$$

Blanket assumption:

all functions under discussion are a.n.n.

<u>Defn</u>

let $g(n)$ be a function.

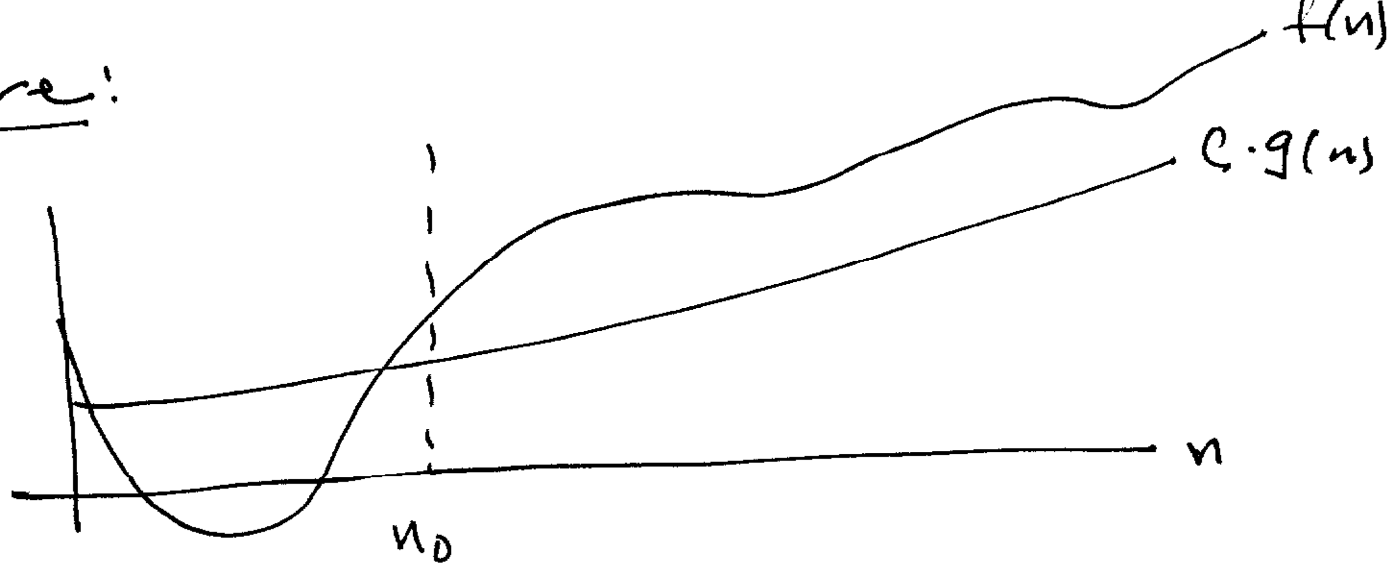
$$\Omega(g(n)) = \{ f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c \cdot g(n) \leq f(n) \}$$

<u>we say</u>: $g(n)$ is an <u>asymptotic lower bound</u> for $f(n)$.

<u>or</u>: $f(n)$ is <u>asymptotically bounded below</u> by $g(n)$

<u>notation</u>: $f(n) = \Omega(g(n))$

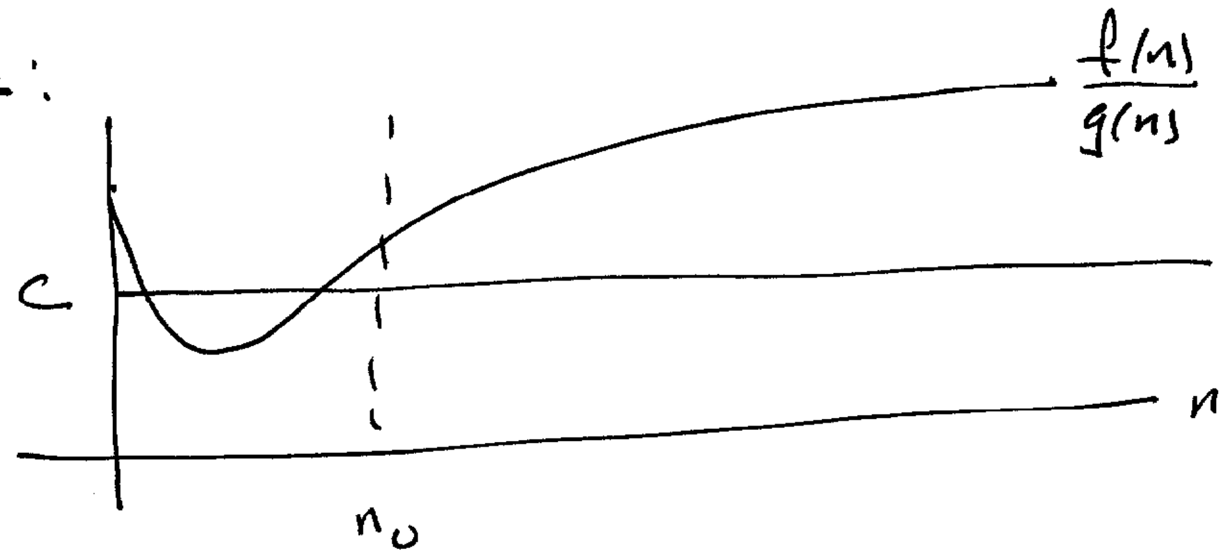
<u>Picture</u>:



or if $g(n)$ is a.p. we have

$$0 \le c \le \frac{f(n)}{g(n)}$$

Picture:



Theorem

$$f(n) = O(g(n)) \overset{\Rightarrow}{\underset{\Leftarrow}{\text{iff}}} g(n) = \Omega(f(n)).$$

Proof:

we prove $\Rightarrow$ only, leaving $\Leftarrow$ as an exercise.

If $f(n) = O(g(n))$ then there exist positive c_1, n_1 s.t. for all $n \geq n_1$:

$$(1) \qquad 0 \leq f(n) \leq c_1 \cdot g(n) \,,$$

we must show that $g(n) = \Omega(f(n))$, i.e. that there exist pos. c_2, n_2 s.t. for all $n \geq n_2$:

$$(2) \qquad 0 \leq c_2 \cdot f(n) \leq g(n) \,,$$

Let $c_2 = \frac{1}{c_1}$ and $n_2 = n_1$.

This proves $g(n) = \Omega(f(n))$. $\blacksquare$