cse 102 10-7-21

Induction Proofs

Goal: Prove stmts of the form

$$\forall n \geq n_0 : P(n)$$

where $P(n)$ is a propositional function

$P : \mathbb{Z} \to \{\text{true}, \text{false}\}$

or a subset of $\mathbb{Z}$, usually $\{0, 1, 2, \ldots\}$ or $\{1, 2, 3, \ldots\}$

Have 2 steps

I. Base: Prove $P(n_0)$ is true

II a. Induction step:

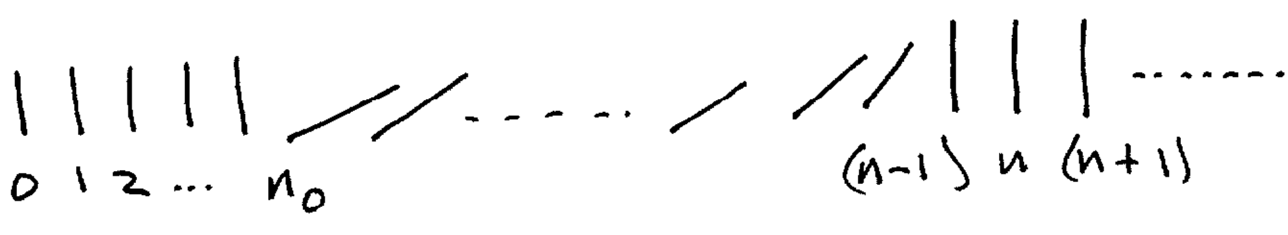
$$\text{Prove } \forall n \geq n_0 : \underbrace{P(n)}_{\text{induction hyp.}} \rightarrow P(n+1)$$

when I & IIa are done, we conclude

$$\forall n \geq n_0 : P(n)$$

is true

Domino analogy:



$P(n)$ = "the n^{th} domino falls"

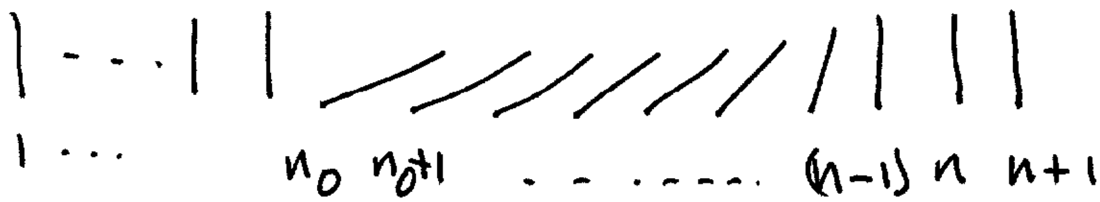
Variations on II:

IIb. Prove $\forall n > n_0 : \underbrace{P(n-1)}_{\text{ind. hyp.}} \to P(n)$

IIa & IIb are called weak induction or the 1^{st} principle of mathematical induction. (PMI)

IIc & IId below are strong induction or the 2^{nd} P.M.I.

IIc: show $\forall n \geq n_0 : \underbrace{\underbrace{P(n_0) \wedge P(n_0+1) \wedge \cdots \wedge P(n)}_{(\forall k \leq n : P(k))}}_{\text{induction hypothesis}} \to \underbrace{P(n+1)}_{\text{induction conclusion}}$



II d. show $\forall n > n_0 : \underbrace{\underbrace{P(n_0) \wedge \dots \wedge P(n-1)}_{\forall k<n : P(k)}}_{\text{ind. hyp.}} \rightarrow P(n)$

same conclusion :

$$\forall n \geq n_0 : P(n)$$

is true.

Ex. $\forall n \geq 1$: $\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$ ← $P(n)$

Proof.

I. $P(1)$ says $\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$

i.e. $1^2 = \frac{1\cdot 2\cdot 3}{6}$, i.e. $1=1$ ✓

II a. show $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$.

Let $n \geq 1$ be chosen arbitrarily.

Assume $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

we must show $\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}$.

so $\sum_{i=1}^{n+1} i^2 = \left(\sum_{i=1}^{n} i^2\right) + (n+1)^2$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \quad \left\{ \begin{array}{l} \text{by the} \\ \text{induction} \\ \text{hypothesis.} \end{array} \right.$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)\left[n(2n+1) + 6(n+1)\right]}{6}$$

$$= \frac{(n+1)\left[2n^2+n+6n+6\right]}{6}$$

$$= \frac{(n+1)(2n^2+7n+6)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}.$$

The result follows for all $n \geq 1$ by the 1^{st} P.M.I. ∎

Ex. let $x \in \mathbb{R}$, $x \neq 1$.

$$\forall n \geq 0 \quad : \quad \boxed{\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}} \leftarrow P(n)$$

Proof:

I. $P(0)$ says $\sum_{i=0}^{0} x^i = \frac{x^1 - 1}{x-1}$, i.e. $1=1$ ✓

II b. show $\forall n > 0 : P(n-1) \Rightarrow P(n)$.

let $n > 0$ be arbitrary. assume

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x-1}.$$

must show that

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}.$$

so

$$\sum_{i=0}^{n} x^i = \left(\sum_{i=0}^{n-1} x^i\right) + x^n$$

$$= \left(\frac{x^n - 1}{x - 1}\right) + x^n \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$

$$= \frac{x^n - 1 + x^n(x-1)}{x - 1}$$

$$= \frac{\cancel{x^n} - 1 + x^{n+1} - \cancel{x^n}}{x - 1}$$

$$= \frac{x^{n+1} - 1}{x - 1}.$$

result follows by 1^{st} P.M.I. ■

Always state

- let $n > n_0$ or $n \geq n_0$
- ind. hyp. explicitely
- ind. conc. explicitely
- the point(s) where the ind. hyp. is used.

Exercises

- re-do 1st ex. using II b
- " 2nd " " II a
- $\forall n \geq 1 : \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$
- $\forall n \geq 1 : \sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

Ex. Define $T(n)$ by

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1 & \text{if } n \geq 2 . \end{cases}$$

n	1	2	3	4	5	6	7	8	...
$T(n)$	0	1	1	2	2	2	2	3	...

Prove that: $\forall n \geq 1 : \boxed{T(n) \leq \lg(n)} \leftarrow P(n)$

Proof:

I. $P(1)$ says $T(1) \leq \lg(1)$, i.e. $0 \leq 0$ ✓

II. $\forall n > 1 : P(1) \wedge \cdots \wedge P(n-1) \Rightarrow P(n)$

Let $n > 1$ be arbitrary. Assume, for this n, that $T(k) \leq \lg(k)$ for all k in the range $1 \leq k < n$.

We must show $T(n) \leq \lg(n)$. So

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1 \quad \left\{ \text{by the recurrence for } T(n) \right.$$

$$\leq \lg\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1 \quad \left\{ \begin{array}{l} \text{by the ind. hyp.} \\ \text{with } k = \left\lfloor \frac{n}{2} \right\rfloor \text{. note} \\ \text{that } 1 \leq \left\lfloor \frac{n}{2} \right\rfloor < n . \end{array} \right.$$

$$\le \lg\left(\frac{n}{2}\right) + 1 \quad \left\{ \begin{array}{l} \text{since } \lfloor x \rfloor \le x \\ \text{and } \lg(\cdot) \text{ is} \\ \text{increasing.} \end{array} \right.$$

$$= \lg(n) - \lg(2) + 1$$

$$= \lg(n) - \cancel{1} + \cancel{1}$$

$$= \lg(n)$$

$\therefore$ $T(n) \le \lg(n)$ holds for all $n \ge 1$ by the 2^{nd} P.M.I. $\blacksquare$