## Exercise 13 c:

compare $3^{2^n}$ to $2^{3^n}$.

$$\frac{2^{2^n}}{3^{2^n}} \longrightarrow \infty$$

implies since $e^{\infty} = \infty$

$$\ln\left(\frac{2^{3^n}}{3^{2^n}}\right) = \ln\left(2^{3^n}\right) - \ln\left(3^{2^n}\right)$$

$$= 3^n \cdot \ln(2) - 2^n \cdot \ln(3) \longrightarrow \infty$$

$$\therefore \quad 2^{3^n} = \omega\left(3^{2^n}\right)$$

Defn

we say $f(n)$ is <u>asymptotically equivalent</u> to $g(n)$ iff

$$\lim_{n \to \infty} \left( \frac{f(n)}{g(n)} \right) = 1$$

<u>notation</u>: $f(n) \sim g(n)$.

obviously : $f(n) \sim g(n) \Rightarrow f(n) = \Theta(g(n))$

<u>Exercise</u>

show $f(n) \sim g(n)$ iff

$$f(n) = g(n) + o(g(n))$$

common functions handout

(CLRS 3.2)

## Floors & ceilings

let $x \in \mathbb{R}$ : $\lfloor x \rfloor$ & $\lceil x \rceil$ are the unique integers satisfying

$$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$$

equivalently: if $N \in \mathbb{Z}$, $x \in \mathbb{R}$

(1) $N = \lfloor x \rfloor$ iff $N \leq x < N+1$

(2) $N = \lceil x \rceil$ iff $N-1 < x \leq N$

Lemma 1

let $x \in \mathbb{R}$, $a, b \in \mathbb{Z}$ :

(1) $a \le x < b$ iff $a \le \lfloor x \rfloor < b$

(2) $a < x \le b$ iff $a < \lceil x \rceil \le b$

Proof.

(i) $a \le x \Rightarrow a \le \lfloor x \rfloor$ : amongst all ints $\le x$, $\lfloor x \rfloor$ is the largest.

(ii) $x < b \Rightarrow \lfloor x \rfloor < b$ : since $\lfloor x \rfloor \le x$

(iii) $a \le \lfloor x \rfloor \Rightarrow a \le x$ : since $\lfloor x \rfloor \le x$

(iv) $\lfloor x \rfloor < b \Rightarrow x < b$ : this is the contrapositive of $b \le x \Rightarrow b \le \lfloor x \rfloor$, which is (i)

$\blacksquare$

Exercise: you do (2).

<u>Lemma 2</u> Let $x \in \mathbb{R}$, $m \in \mathbb{Z}^+$.

(1) $\left\lfloor \frac{\lfloor x \rfloor}{m} \right\rfloor = \left\lfloor \frac{x}{m} \right\rfloor$ ✓

(2) $\left\lceil \frac{\lceil x \rceil}{m} \right\rceil = \left\lceil \frac{x}{m} \right\rceil$

<u>Proof of (1)</u>

Let $N = \left\lfloor \frac{\lfloor x \rfloor}{m} \right\rfloor$. Then

$$N \le \frac{\lfloor x \rfloor}{m} < N+1$$

$\therefore \quad mN \le \lfloor x \rfloor < m(N+1)$

$\therefore \quad mN \le x < m(N+1)$ $\left\{ \begin{array}{l} \text{by lemma 1} \\ \text{Part (1)} \end{array} \right.$

$\therefore \quad N \le \frac{x}{m} < N+1$

$\therefore \quad N = \left\lfloor \frac{x}{m} \right\rfloor$ ∎

<u>Exercise:</u> Prove (2).

Lemma 3 : let $a, b, n \in \mathbb{Z}^+$

(1) $\left\lfloor \frac{\lfloor n/a \rfloor}{b} \right\rfloor = \left\lfloor \frac{n}{ab} \right\rfloor$

(2) $\left\lceil \frac{\lceil n/a \rceil}{b} \right\rceil = \left\lceil \frac{n}{ab} \right\rceil$

Proof: let $x = n/a$ in lemma 2. ∎

Logarithms : $x > 0$, $a > 1$, $b > 1$.

Recall $\log_a(x)$ is inverse fcn to $a^x$.

$$x = a^{\log_a x} = \left( b^{\log_b(a)} \right)^{\log_a x}$$

$$\therefore \quad x = b^{\log_b(a) \cdot \log_a(x)}$$

$$\therefore \quad \log_b(x) = \underbrace{\log_b(a)}_{\text{const}} \cdot \log_a(x) \qquad (*)$$

$$\therefore \quad \log_b(n) = \text{const} \cdot \log_a(n)$$

hence $\boxed{\log_b(n) = \Theta(\log_a(n)).}$

Also by $(*)$:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

Also by $(*)$

$$a^{\log_b(x)} = a^{\log_a(x) \cdot \log_b(a)} = \left(a^{\log_a(x)}\right)^{\log_b(a)}$$

$$\therefore \quad \boxed{a^{\log_b(x)} = x^{\log_b(a)}}$$

# Stirling's Formula

Let $n \in \mathbb{Z}^+$. Then

$$\boxed{n! = \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)}$$

## Corollary:

(1) $n! = o(n^n)$

(2) $n! = \omega(b^n)$ for any $b > 0$

(3) $\log(n!) = \Theta(n \log n)$

## Proof of (1)

$$\frac{n!}{n^n} = \frac{\sqrt{2\pi} \cdot \sqrt{n} \cdot \frac{\cancel{n^n}}{e^n} \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)}{\cancel{n^n}}$$

$$= \sqrt{2\pi} \cdot \frac{n^{1/2}}{e^n} \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right) \longrightarrow 0$$

## Proof of (3)

$$\frac{\log(n!)}{n\log n} = \frac{\log\sqrt{2\pi} + \log\sqrt{n} + \log n^n - \log e^n + \log\left(1+\Theta\left(\frac{1}{n}\right)\right)}{n\log n}$$

$$= \frac{\log\sqrt{2\pi}}{n\log n} + \frac{\frac{1}{2}\cancel{\log n}}{n\cancel{\log n}} + \cancel{\frac{n\log n}{n\log n}} - \frac{\cancel{n}\log e}{\cancel{n}\log n} + \frac{\log\left(1+\Theta\left(\frac{1}{n}\right)\right)}{n\log n}$$

$$= \underbrace{\frac{\log\sqrt{2\pi}}{n\log n}}_{\downarrow\ 0} + \underbrace{\frac{1}{2n}}_{\downarrow\ 0} + \underbrace{1}_{\downarrow\ 1} - \underbrace{\frac{\log e}{\log n}}_{\downarrow\ 0} + \underbrace{\frac{\log\left(1+\Theta\left(\frac{1}{n}\right)\right)}{n\log n}}_{\downarrow\ 0}$$

$$\therefore \lim_{n\to\infty} \frac{\log(n!)}{n\log n} = 1 \in (0, \infty).$$

$$\therefore \quad \log(n!) = \Theta(n\log n). \qquad \blacksquare$$

Exercise: Prove (2).

**Exercise:** $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$

**Proof:** recall $\binom{m}{k} = \dfrac{m!}{k!\,(m-k)!}$

$$\binom{2n}{n} = \frac{(2n)!}{n! \cdot n!}$$

$$= \frac{\sqrt{2\pi}\cdot\sqrt{2n}\cdot\left(\frac{2n}{e}\right)^{2n}\cdot\left(1+\Theta\left(\frac{1}{2n}\right)\right)}{\sqrt{2\pi}^{\,2}\cdot\sqrt{n}^{\,2}\cdot\left(\frac{n}{e}\right)^{2n}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\sqrt{2}\cdot\frac{1}{\sqrt{n}}\cdot\frac{\left(\frac{2^{2n}\cdot\cancel{n^{2n}}}{\cancel{e^{2n}}}\right)}{\left(\frac{\cancel{n^{2n}}}{\cancel{e^{2n}}}\right)}\cdot\left[\frac{1+\Theta\left(\frac{1}{2n}\right)}{\left(1+\Theta\left(\frac{1}{n}\right)\right)^2}\right]$$

$$= \frac{\sqrt{2}}{\sqrt{2\pi}}\cdot\frac{4^n}{\sqrt{n}}\cdot\left[\qquad\qquad\right]$$

$$\therefore \quad \frac{\binom{2n}{n}}{\frac{4^n}{\sqrt{n}}} = \frac{\sqrt{\cancel{2}}}{\sqrt{\cancel{2}\pi}} \cdot \Big[ \qquad \Big] \longrightarrow \frac{1}{\sqrt{\pi}}.$$

since $0 < \frac{1}{\sqrt{\pi}} < \infty$ we have

$$\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right). \qquad \blacksquare$$