cse 102 10-19-21

- Exam Thursday !!
- hw3 due tonight.

more on Iteration method.

<u>Ex.</u>

$$T(n) = \begin{cases} c & 1 \le n < n_0 \\ T(\lfloor \frac{n}{2} \rfloor) + d & n \ge n_0 \end{cases}$$

$$T(\text{☺}) = d + T\left(\left\lfloor \frac{\text{☺}}{2} \right\rfloor\right)$$

$$\begin{aligned} T(n) &= d + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \\ &= d + d + T\left(\left\lfloor \frac{\lfloor n/2 \rfloor}{2} \right\rfloor\right) \\ &= 2d + T\left(\left\lfloor \frac{n}{2^2} \right\rfloor\right) \end{aligned}$$

$$\vdots$$

$$= kd + T\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right)$$

The recurrence terminates at the first (i.e. smallest) k for which

$$\left\lfloor \frac{n}{2^k} \right\rfloor < n_0$$

i.e $$\frac{n}{2^k} < n_0$$

$$\therefore \quad \frac{n}{n_0} < 2^k$$

$$\therefore \quad k-1 \leq \lg\left(\frac{n}{n_0}\right) < k$$

$$\therefore \quad k-1 = \left\lfloor \lg\left(\frac{n}{n_0}\right) \right\rfloor$$

$$\therefore \quad \boxed{k = \left\lfloor \lg\left(\frac{n}{n_0}\right) \right\rfloor + 1}$$

for this k we have:

$$\boxed{T(n) = d\left(\left\lfloor \lg\left(\frac{n}{n_0}\right)\right\rfloor + 1\right) + c}$$ ← exact soln.

so

$$T(n) = d\lfloor \lg n - \lg n_0 \rfloor + d + c$$

$$= \Theta(d \lg n - d\lg n_0 + d + c)$$

$$\boxed{T(n) = \Theta(\log n)}$$ ← asymptotic soln.

Read next example!

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor \frac{n}{2} \rfloor) + n^2 & n \geq 2 \end{cases}$$

Master method

<u>Goal</u>: find asymptotic solutions to recurrences of form

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

It is required that $a \geq 1$, $b > 1$ and $f(n)$ asymptotically positive.

we get

$$T(n) = \Theta(\text{ something })$$

depends only on a, b and on $\Theta(f(n))$.

Master-Theorem

let $a \geq 1$, $b > 1$ and $f(n)$ a.p.
define $T(n) = aT(\frac{n}{b}) + f(n)$. we have 3 cases.

(1) if $f(n) = O(n^{\log_b(a) - \varepsilon})$ for some $\varepsilon > 0$,

then $\boxed{T(n) = \Theta(n^{\log_b(a)})}$

(2) if $f(n) = \Theta(n^{\log_b(a)})$, then

$$\boxed{T(n) = \Theta(n^{\log_b(a)} \cdot \log n)}$$

(3) if $f(n) = \Omega(n^{\log_b(a) + \varepsilon})$ for some $\varepsilon > 0$

and $\boxed{a f(\frac{n}{b}) \leq c f(n) \text{ for some } 0 < c < 1 \text{ and all suff. large } n}$, then

$$\boxed{T(n) = \Theta(f(n))}.$$

Rule

we compare $f(n)$ to $n^{\log b(a)}$

(1) $f(n)$ 'loses' by Polynomial factor n^{ε}

(2) $f(n)$ and $n^{\log b(a)}$ tie, we get factor of log(n)

(3) $f(n)$ 'wins' by Poly factor n^{ε}.

also have regularity condition

Ex. $T(n) = 8T\left(\frac{n}{2}\right) + n^3$.

compare: n^3 to $n^{\log_2 8} = n^3$ (tie.)

by case (2): $\boxed{T(n) = \Theta(n^3 \cdot \log(n))}$

Ex. $T(n) = 5T\left(\frac{n}{4}\right) + n$

Comp: n to $n^{\log_4(5)}$

let $\varepsilon = \log_4(5) - 1$. Then since $\log_4(5) > 1$, we have $\varepsilon > 0$. Also

$$n = O(n^1) = O\left(n^{\log_4(5) - \varepsilon}\right).$$

by case (1): $\boxed{T(n) = \Theta\left(n^{\log_4(5)}\right)}$

Ex. $T(n) = 5T\left(\frac{n}{4}\right) + n^2$

comp n^2 to $n^{\log_4(5)}$

note: $5 < 16 \Rightarrow \log_4(5) < \log_4(16) = 2$

let $\varepsilon = 2 - \log_4(5)$. Then

$\varepsilon > 0$ and $2 = \log_4(5) + \varepsilon$. Thus

$$n^2 = \Omega(n^2) = \Omega\left(n^{\log_4(5)+\varepsilon}\right).$$

need regularity cond: $af\left(\frac{n}{b}\right) \le c\,f(n)$ for some $0 < c < 1$, all suff. large n.

i.e. $5\left(\frac{n}{4}\right)^2 \le c n^2$

i.e. $\frac{5}{16} n^2 \le c n^2$

i.e. $\frac{5}{16} \le c < 1$

such a c exist since $\frac{5}{16} < 1$.

Pick any $c \in \left[\frac{5}{16}, 1\right)$. Thus

$$\boxed{T(n) = \Theta\left(n^2\right)}$$

Ex. $T(n) = 8T(\frac{n}{2}) + \underbrace{10n^3 + 15n^2 - n^{1.5} + n\log n + 1}_{\text{replace by } n^3}$

1^{st} example: $T(n) = 8T(\frac{n}{2}) + n^3$

$T(n) = \Theta(n^3 \cdot \log n)$.

Rmk:

when $f(n)$ is a Polynomial, everything is easy. let

(1) $\varepsilon = \log_b(a) - \deg(f)$

(2) tie

(3) $\varepsilon = \deg(f) - \log_b(a)$

reg cond. always works when $f(n)$ is a Poly.

Exercise

Prove that if $f(n)$ is a polynomial, say $f(n) = \Theta(n^d)$, and $d > \log_b(a)$, then case (3) holds and reg. cond. succeeds.

Ex. $T(n) = T(\lfloor \frac{n}{2} \rfloor) + 2T(\lceil \frac{n}{2} \rceil) + \log(n!)$

simplify:

$$T(n) = 3T\left(\frac{n}{2}\right) + n\log n$$

Comp. $n\log n$ to $n^{\log_2(3)}$

let $\varepsilon = \frac{1}{2}\left(\log_2(3) - 1\right)$. Then $\varepsilon > 0$

and $2\varepsilon = \log_2(3) - 1$.

i. $1+\varepsilon = \log_2(3) - \varepsilon$.

observe

$$\frac{n \log n}{n^{1+\varepsilon}} = \frac{\cancel{n} \log n}{\cancel{n} \cdot n^{\varepsilon}} = \frac{\log n}{n^{\varepsilon}} \longrightarrow 0$$

$$\therefore \; n \log n = o(n^{1+\varepsilon}) \subseteq O(n^{1+\varepsilon})$$

$$\therefore \; n \log n = O(n^{\log_2(3) - \varepsilon})$$

By case (1) : $\boxed{T(n) = \Theta(n^{\log_2(3)})}$

<u>note</u>. $\varepsilon = \log_2(3) - 1$ will not work!!

we would have $1 = \log_2(3) - \varepsilon$, so

$$\frac{n \log n}{n^{\log_2(3) - \varepsilon}} = \frac{\cancel{n} \log n}{\cancel{n}^{1}} = \log n \longrightarrow \infty.$$

and so $n\log n = \omega\left(n^{\log_2(3)-\varepsilon}\right)$,

hence $n\log n \neq O\left(n^{\log_2(3)-\varepsilon}\right)$.

<u>Ex.</u> $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}$

master thm does not apply !!

since $n^{\log_2 2} = n^1$ wins to $\frac{n}{\log n}$,

but not by a poly factor.