CSE 102 10-15-21

Supplemental lecture

Substitution method

Ex. $T(n) = \begin{cases} 2 & 1 \le n < 3 \\ 3T(\lfloor \frac{n}{3} \rfloor) + n & n \ge 3 \end{cases}$

Guess: $T(n) = O(n \log n)$.

choose $\log(\cdot) = \log_3(\cdot)$

must find pos. c, n_0 s.t.

$$\forall n \ge n_0 : \boxed{T(n) \le c \cdot n \log n} \leftarrow P(n)$$

note: $T(1) = 2 \nleq 0 = c \cdot 1 \cdot \log(1)$

$\therefore$ can't start at 1, i.e. $n_0 \ne 1$.
let $\boxed{n_0 = 2}$

let n_1 be the highest base case.

<u>mimic IHd</u>: $\forall n > n_1 : P(2) \wedge \dots \wedge P(n-1) \rightarrow P(n)$

let $n > n_1$. Assume for all k in range $2 \le k < n$ that $T(k) \le c k \log(k)$. We must show $T(n) \le c n \log(n)$. So

$$T(n) = 3T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n$$

$$\le 3c\left\lfloor \frac{n}{3} \right\rfloor \log\left\lfloor \frac{n}{3} \right\rfloor + n \quad \left\{ \begin{array}{l} \text{by ind. hyp.} \\ k = \lfloor n/3 \rfloor \end{array} \right.$$

<u>note</u>: this requires that

$$2 \le \left\lfloor \frac{n}{3} \right\rfloor < n$$

$$2 \le \frac{n}{3} < n$$

$6 \le n$ i.e. $n > 5$

let $\boxed{n_1 = 5}$

$$\leq \cancel{3}c\cdot\left(\frac{n}{\cancel{3}}\right)\log\left(\frac{n}{3}\right)+n \quad \left\{ \begin{array}{l} \text{since} \\ \lfloor x \rfloor \leq x \end{array} \right.$$

$$= cn(\log n - \log 3) + n$$

$$= cn\log n - cn + n \underset{\substack{\uparrow \\ \text{Want}}}{\leq} cn\log n$$

i.e. $-cn + n \leq 0$

$$n \leq cn$$

$$1 \leq c$$

what aboot base?

base cases: $n = \underset{\substack{\uparrow \\ n_0}}{2}, 3, 4, \underset{\substack{\uparrow \\ n_1}}{5}$

need: $T(2) \leq c\cdot 2\log 2$

$T(3) \leq c\cdot 3\log 3$

$T(4) \leq c\, 4\log 4$

$T(5) \leq c\, 5\log 5$

check:

$T(2)=2,\ T(3)=9,\ T(4)=10,\ T(5)=11$

need

$$c \ge \frac{2}{2\log 2}$$

$$c \ge \frac{9}{3\log 3}$$

$$c \ge \frac{10}{4\log 4}$$

$$c \ge \frac{11}{5\log 5}$$

check

$$\max\left(1, \frac{2}{2\log 2}, \frac{9}{3\log 3}, \frac{10}{4\log 4}, \frac{11}{5\log 5}\right) = 3$$

max

pick $\boxed{c = 3}$

Proof that: $\forall n \geq 2: \boxed{T(n) \leq 3n\log(n)}$

I. check $T(2) \leq 3 \cdot 2\log 2$ ✓

$T(3) \leq 3 \cdot 3\log 3$ ✓

$T(4) \leq 3 \cdot 4\log 4$ ✓

$T(5) \leq 3 \cdot 5\log 5$ ✓

IId. $\forall n > 5: P(2) \wedge \ldots \wedge P(n-1) \Rightarrow P(n)$.

let $n > 5$ be arbitrary. Assume for all k in range $2 \leq k < n$ that

$$T(k) \leq 3k\log k.$$

must show $T(n) \leq 3n\log n$. so,

$$T(n) = 3T\left(\left\lfloor\frac{n}{3}\right\rfloor\right) + n$$

$$\leq 3 \cdot 3\left\lfloor\frac{n}{3}\right\rfloor\log\left\lfloor\frac{n}{3}\right\rfloor + n$$ by ind. hyp. with $k = \left\lfloor\frac{n}{3}\right\rfloor$

$$\leq 9 \cdot \left(\frac{n}{3}\right)\left(\log \frac{n}{3}\right) + n \quad \left\{ \begin{array}{l} \text{since} \\ \lfloor x \rfloor \leq x \end{array} \right.$$

$$= 3n(\log n - 1) + n$$

$$= 3n \log n - 3n + n$$

$$= 3n \log n - 2n$$

$$\leq 3n \log n . \quad \left\{ \begin{array}{l} \text{since} \\ -2n \leq 0 \end{array} \right.$$

we have $\forall n \geq 2 : T(n) \leq 3n \log n$

$\therefore \; T(n) = O(n \log n)$.

<u>Exercise</u> (same recurrence)
determine pos. c, n_0 s.t. for all $n \geq n_0$.

$$T(n) \geq c n \log n .$$

Hence $T(n) = \Omega(n \log n)$, $\therefore T(n) = \Theta(n \log n)$.

Ex. $T(n) = \begin{cases} 1 & n = 1 \\ 2T(\lfloor \frac{n}{2} \rfloor) + 1 & n \geq 2 \end{cases}$

Guess: $T(n) = O(n)$.

want pos. c, n_0 s.t.

$\forall n \geq n_0 : T(n) \leq cn$.

mimic II d.

let $n > n_0$. Assume for all k s.t.

$n_0 \leq k < n$ that $T(k) \leq ck$. must

show $T(n) \leq cn$. so

$$T(n) = 2T(\lfloor \tfrac{n}{2} \rfloor) + 1$$

$$\leq 2 \cdot c \lfloor \tfrac{n}{2} \rfloor + 1 \quad \begin{cases} \text{by ind. hyp.} \\ \text{with } k = \lfloor \frac{n}{2} \rfloor \end{cases}$$

$$\leq \cancel{2} c \left(\tfrac{n}{\cancel{2}}\right) + 1$$

$$= cn + 1 \le cn$$

$$\uparrow$$

want.

we want: $1 \le 0$

so this way won't work.. instead we show there exist pos. c, n_0, b such that

$$\forall n \geq n_0 : T(n) \le \underbrace{cn - b}_{\text{in } O(n)}$$

Trick: subtract a lower order term.

mimic II d.

let $n > n_0$. Assume for all k: $n_0 \le k < n$ that $T(k) \le ck - b$. must show $T(n) \le cn - b$.

$$T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1$$

$$\leq 2\left(c\left\lfloor \frac{n}{2} \right\rfloor - b\right) + 1 \quad \begin{cases} \text{by ind. hyp.} \\ k = \left\lfloor \frac{n}{2} \right\rfloor . \end{cases}$$

need. $n_0 \leq \left\lfloor \frac{n}{2} \right\rfloor < n$

~~suff.~~ to choose $\boxed{n_0 = 1}$. Then

$$n > 1 \Rightarrow n \geq 2$$
$$\Rightarrow \frac{n}{2} \geq 1$$
$$\Rightarrow \left\lfloor \frac{n}{2} \right\rfloor \geq 1 .$$

$$= 2c\left\lfloor \frac{n}{2} \right\rfloor - 2b + 1$$

$$\leq \cancel{2}c\left(\frac{n}{\cancel{2}}\right) - 2b + 1$$

$$= cn - 2b + 1 \leq cn - b$$

want

i.e. $-2b + 1 \leq -b$

$$1 \leq b \Rightarrow \text{Pick } \boxed{b = 1}$$

mimic base:

$$T(1) \le c \cdot 1 - 1$$

i.e. $1 \le c - 1$

$$2 \le c \Rightarrow \text{Pick } \boxed{c = 2}$$

Exercise

define $T(n) = \begin{cases} 1 & n = 1 \\ 2T(\lfloor \frac{n}{2} \rfloor) + 1 & n \ge 2 \end{cases}$.

show that for all $n \ge 1$ we have

$$T(n) \le 2n - 1$$

Hence $T(n) = O(n)$, since $2n - 1 = O(n)$.

Recursion Tree/Iteration

Ex.

$$T(n) = \begin{cases} 1 & 1 \le n < 3 \\ 2T(\lfloor \frac{n}{3} \rfloor) + n & n \ge 3 \end{cases}$$

simplify, and lose information: write

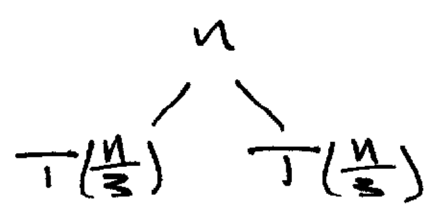
$$T(n) = 2T\left(\frac{n}{3}\right) + n = n + T\left(\frac{n}{3}\right) + T\left(\frac{n}{3}\right)$$

Draw a sequence of trees.
each node in each tree one recursive invocation of $T(\ )$.

0^{th} Tree

$T(n)$

1^{st} tree



continue in handout

⋮

k^{th} tree has 2^i nodes at depth i, each of value $\left(\frac{n}{3^i}\right)$.

recurrence terminates when

$$\frac{n}{3^k} = 1$$

$$\therefore \quad n = 3^k$$

$$\therefore \quad \boxed{\log_3(n) = k}$$

at this k, we have $T\left(\frac{n}{3^k}\right) = 1$

Adding all these nodes gives

$$T(n) = \sum_{i=0}^{k-1} 2^i \cdot \left(\frac{n}{3^i}\right) + 2^k \cdot T(1)$$

$$= n \cdot \sum_{i=0}^{k-1} \left(\frac{2}{3}\right)^i + 2^{\log_3(n)}$$

$$= n\left(\frac{1-(2/3)^k}{1-(2/3)}\right) + n^{\log_3(2)}$$

$$= 3n\left(1-\left(\tfrac{2}{3}\right)^{\log_3(n)}\right) + n^{\log_3(2)}$$

$$= \Theta(n) + \Theta(n^{\log_3(2)})$$

Guess: $T(n) = \Theta(n)$.

<u>Exercise</u>:
Use substitution to show

- $T(n) = O(n)$
- $T(n) = \Omega(n)$.

Iteration method

Ex. $$T(n) = \begin{cases} 1 & 1 \le n < 3 \\ 2T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n & n \ge 3 \end{cases}$$

$$T(☺) = ☺ + 2T\left(\left\lfloor \frac{☺}{3} \right\rfloor\right)$$

$$\therefore \; T(n) = n + 2T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) \qquad (1)$$

$$= n + 2\left(\left\lfloor \frac{n}{3} \right\rfloor + 2T\left(\left\lfloor \frac{\lfloor n/3 \rfloor}{3} \right\rfloor\right)\right)$$

$$= n + 2\left\lfloor \frac{n}{3} \right\rfloor + 2^2 \cdot T\left(\left\lfloor \frac{n}{3^2} \right\rfloor\right) \qquad (2)$$

$$= n + 2\left\lfloor \frac{n}{3} \right\rfloor + 2^2\left(\left\lfloor \frac{n}{3^2} \right\rfloor + 2T\left(\left\lfloor \frac{\lfloor n/3^2 \rfloor}{3} \right\rfloor\right)\right)$$

$$= n + 2\left\lfloor \frac{n}{3} \right\rfloor + 2^2\left\lfloor \frac{n}{3^2} \right\rfloor + 2^3 T\left(\left\lfloor \frac{n}{3^3} \right\rfloor\right) \qquad (3)$$

$$\vdots$$

$$= n + 2\left\lfloor \frac{n}{3} \right\rfloor + 2^2\left\lfloor \frac{n}{3^2} \right\rfloor + 2^3\left\lfloor \frac{n}{3^3} \right\rfloor + 2^4 T\left(\left\lfloor \frac{n}{3^4} \right\rfloor\right) \qquad (4)$$

$$\vdots$$

$$= \sum_{i=0}^{k-1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor + 2^k T\left(\left\lfloor \frac{n}{3^k} \right\rfloor\right) \qquad (k)$$

recurrence terminates when

$$1 \le \left\lfloor \frac{n}{3^k} \right\rfloor < 3$$

$$\therefore \quad 1 \le \frac{n}{3^k} < 3$$

$$3^k \le n < 3^{k+1}$$

$$k \le \log_3(n) < k+1$$

$$\therefore \quad \boxed{k = \lfloor \log_3(n) \rfloor}$$

so for this k: $T\left(\left\lfloor \frac{n}{3^k} \right\rfloor\right) = 1$, and

$$T(n) = \sum_{i=0}^{k-1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor + 2^k$$

first find asymp. up. bound :

$$T(n) = \sum_{i=0}^{k-1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor + 2^{\lfloor \log_3(n) \rfloor}$$

$$\leq n\left(\sum_{i=0}^{k-1} \left(\frac{2}{3}\right)^i\right) + 2^{\log_3(n)}$$

$$\leq n\left(\sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i\right) + n^{\log_3(2)}$$

$$= n\left(\frac{1}{1-2/3}\right) + n^{\log_3(2)}$$

$$= 3n + n^{\log_3(2)} = O(n)$$

$\therefore \quad T(n) = O(n)$,

Also

$$T(n) = 2T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n \geq n = \Omega(n)$$

$\therefore T(n) = \Omega(n) \quad \therefore \boxed{T(n) = \Theta(n)}$

Ex. $T(n) = \begin{cases} 5 & n = 0,1 \\ T(n-2) + n & n \geq 2 \end{cases}$

$$T(n) = n + T(n-2)$$

$$= n + (n-2) + T(n-2\cdot 2)$$

$$= n + (n-2) + (n-2\cdot 2) + T(n-2\cdot 3)$$

$$\vdots$$

$$= \sum_{i=0}^{k-1} (n-2\cdot i) + T(n-2\cdot k)$$

Terminate when $n-2k=0$ or $n-2k=1$, i.e.

$$0 \leq n-2k < 2$$

$$2k \leq n < 2k+2$$

$$k \leq \frac{n}{2} < k+1$$

$$\therefore \boxed{k = \left\lfloor \frac{n}{2} \right\rfloor}.$$

for this k:

$$T(n) = \sum_{i=0}^{k-1} (n-2i) + 5$$

$$= \sum_{i=0}^{k-1} n - 2\sum_{i=\cancel{0}\,1}^{k-1} i + 5$$

$$= kn - \cancel{2} \cdot \frac{k(k-1)}{\cancel{2}} + 5$$

$$= kn - k(k-1) + 5$$

$$\therefore\ T(n) = \left\lfloor \frac{n}{2} \right\rfloor \cdot n - \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + 5$$

$$\therefore\ T(n) = \Theta(n^2).$$

Ex. $$T(n) = \begin{cases} 0 & n=1 \\ T\left(\left\lfloor \frac{n}{2} \right\rfloor\right)+1 & n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= 1 + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \\ &= 1 + 1 + T\left(\left\lfloor \frac{\lfloor n/2 \rfloor}{2} \right\rfloor\right) = 2 + T\left(\left\lfloor \frac{n}{2^2} \right\rfloor\right) \\ &= 2 + 1 + T\left(\left\lfloor \frac{\lfloor n/2^2 \rfloor}{2} \right\rfloor\right) = 3 + T\left(\left\lfloor \frac{n}{2^3} \right\rfloor\right) \\ &\ \vdots \\ &= k + T\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right) \end{aligned}$$

Bottom out when $\left\lfloor \frac{n}{2^k} \right\rfloor = 1$, i.e.

$$1 \leq \left\lfloor \frac{n}{2^k} \right\rfloor < 2$$

$$1 \leq \frac{n}{2^k} < 2$$

$$2^k \leq n < 2^{k+1}$$

$$k \leq \log_2(n) < k+1$$

$$\therefore \boxed{k = \lfloor \log_2(n) \rfloor}$$

for this k : $T\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right) = 0$

$$T(n) = k + 0$$

$\therefore$ $\boxed{T(n) = \lfloor \lg(n) \rfloor}$ $\leftarrow$ exact soln.

$\lg(\cdot) = \log_2(\cdot)$

$\therefore$ $\boxed{T(n) = \Theta(\log n)}$ $\leftarrow$ asymptotic soln.

<u>Exercise</u>

check directly that $T(n) = \lfloor \lg(n) \rfloor$ satisfies recurrence.

<u>Exercise</u>

let $S(n) = \begin{cases} 0 & n = 1 \\ S(\lceil \frac{n}{2} \rceil) + 1 & n \geq 2. \end{cases}$

Show $S(n) = \lceil \lg(n) \rceil$, hence $S(n) = \Theta(\log n)$