cse 102 10-14-21

Induction Principle

1PMI :

for any Prop. fcn. $P: \mathbb{Z}^+ \to \{\text{false}, \text{true}\}$,
the following sentence is true

$$\left[P(1) \land \left(\forall n > 1 : P(n-1) \to P(n)\right)\right] \to \forall n \geq 1 : P(n)$$

2PMI :

for any Prop. fcn. $P: \mathbb{Z}^+ \to \{\text{false}, \text{true}\}$
the following sentence is true.

$$\left[P(1) \land \left(\forall n > 1 : \left(\forall k < n : P(k)\right) \to P(n)\right)\right] \to \forall n \geq 1 : P(n)$$

Well ordering Property of $\mathbb{Z}^+$ (WOP)

Any non-empty subset of $\mathbb{Z}^+$ contains a least element.

Thm 1

$$\text{WOP} \Longrightarrow \text{1-PMI}$$

Proof.

Let $P : \mathbb{Z}^+ \to \{F, T\}$ be a Prop. fcn. Assume $P(1)$ and $\forall n > 1 : P(n-1) \to P(n)$ are both true. Let

$$S = \{ n \in \mathbb{Z}^+ \mid P(n) \text{ is false} \},$$

so $S \subseteq \mathbb{Z}^+$. Its suff. to prove $S = \emptyset$, for then $\forall n \geq 1 : P(n)$ is true.

Assume, to get a $\times$, that $S \neq \emptyset$.

By the WOP, S contains a least element, call it m. Since $m \in S$, $P(m)$ is false. Since $P(1)$ is true, $m \neq 1$, hence $m > 1$. Also since m is least in S, we have $P(m-1)$ is true. Since

$$\forall n > 1 : P(n-1) \to P(n)$$

is true, we have, for $n = m$ that

$$P(m-1) \to P(m)$$

is true. It follows that $P(m)$ must also be true. This contradicts that $P(m)$ is false.

we conclude our assumotion was false, hence $S=\phi$, and

$$\forall n \geq 1 : P(n)$$

is true. $\blacksquare$

$$\begin{tikzcd}
& \mathrm{WOP} \arrow[dl, bend right, "1"'] \arrow[dr, bend left, "2"] & \\
\mathrm{1PMI} \arrow[ur, bend left, "\text{exercise}"] \arrow[rr, bend right, "\text{hw2 \#6 exercise}"] & & \mathrm{2PMI} \arrow[ul, bend left, "3"] \arrow[ll, bend left, "\text{exercise}"]
\end{tikzcd}$$

Recurrence Relations

Ex.

$$T(n) = \begin{cases} c & n = 1 \\ T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + dn & n \geq 2 \end{cases}$$

- explicit solution:
  numerical fcn. satisfying the above relation
- asymptotic solution:
  asymtotic growth rate of explicit solution.

methods

- substitution method
- Recursion tree/Iteration method
- Master method

Substitution method

Ex.

$$T(n) = \begin{cases} 2 & 1 \le n < 3 \\ 3T(\lfloor \frac{n}{3} \rfloor) + n & n \ge 3 \end{cases}$$

Guess: $T(n) = O(n \log(n))$.

must find pos. c, n_0 s.t.

$$T(n) \le c\, n \log(n)$$

for all $n \ge n_0$.

Base cases:

X. $n=1$: $T(1) \le c \cdot 1 \cdot \log(1)$, $2 \le c \cdot 0$ - X.

so false in this case!!

$n=2$: $T(2) \le c \cdot 2 \cdot \log(2)$, $2 \le c \cdot 2 \cdot \log(2)$

i.e. $c \ge \frac{1}{\log(2)}$

$n=3$: $T(3) \le c \cdot 3\log 3$, $c \ge \dfrac{T(3)}{3\log(3)}$

$n=4$: $T(4) \le c \cdot 4\log 4$, $c \ge \dfrac{T(4)}{4\log(4)}$

$n=5$: $T(5) \le c\,5\log 5$, $c \ge \dfrac{T(5)}{5\log(5)}$

Pick $\boxed{\log = \log_3}$

Induction: $\forall n > 5 : P(2) \wedge \ldots \wedge P(n-1) \rightarrow P(n)$.

let $n > 5$. Assume for all k in range $2 \le k < n$ that $T(k) \le c \cdot k \log(k)$.

must show $T(n) \le c\,n \log(n)$. so

$$T(n) = 3T\left(\left\lfloor \tfrac{n}{3} \right\rfloor\right) + n$$

$$\le 3 \cdot c \cdot \left\lfloor \tfrac{n}{3} \right\rfloor \cdot \log \left\lfloor \tfrac{n}{3} \right\rfloor + n$$

by ind. hyp. with $k = \lfloor \frac{n}{3} \rfloor$, since $2 \le \lfloor \frac{n}{3} \rfloor < n$

continue on handout

⋮

more in supplemental lecture