CSE 102 10-12-21

Ex. Define

$$T(n)=\begin{cases} 1 & \text{if } 1 \le n \le 2 \\ 4T\left(\left\lfloor \frac{n}{3} \right\rfloor\right)+n & \text{if } n \ge 3 \end{cases}$$

Multiple base cases:

lowest base case highest base case

- Base: $P(1), P(2), \ldots, P(n_0)$

Induction (weak): $\forall n > n_0 : P(n-1) \Rightarrow P(n)$

- Induction (strong): $\forall n > n_0 : P(1) \wedge \cdots \wedge P(n-1) \rightarrow P(n)$

Conclusion: $\forall n \ge 1 : P(n)$

Prove that $\forall n \geq 1$: $\boxed{T(n) \leq n^2}$ $\leftarrow P(n)$

hence $T(n) = O(n^2)$.

Proof: 2 base cases $n = 1, 2$

I. $n=1$. $P(1)$ says $T(1) \leq 1^2$, i.e. $1 \leq 1$ ✓

$n=2$. $P(2)$ says $T(2) \leq 2^2$, i.e. $1 \leq 4$ ✓

II. $\forall n > 2 : P(1) \wedge \dots \wedge P(n-1) \rightarrow P(n)$.

Let $n > 2$ be chosen arbitrarily.

Assume for all k in range $1 \leq k < n$

that $T(k) \leq k^2$. We must show

$T(n) \leq n^2$. So

$$\begin{aligned} T(n) &= 4T\left(\left\lfloor \tfrac{n}{3} \right\rfloor\right) + n \\ &\leq 4\left\lfloor \tfrac{n}{3} \right\rfloor^2 + n \leftarrow \end{aligned}$$

by the ind hyp. $k = \left\lfloor \frac{n}{3} \right\rfloor$

note: $1 \leq \left\lfloor \frac{n}{3} \right\rfloor < n$

$$\le 4\left(\frac{n}{3}\right)^2 + n \quad \{\text{since } \lfloor x \rfloor \le x$$

$$= \frac{4}{9}n^2 + n$$

$$\le n^2 \quad \checkmark$$

$\uparrow$

why ?

$$n > 2 \Rightarrow \frac{9}{5} \le n \Rightarrow n \le \frac{5}{9}n^2 \Rightarrow \frac{4}{9}n^2 + n \le n^2$$

∎

why 2 base cases ?

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...
$\lfloor \frac{n}{3} \rfloor$	•	•	1	1	1	2	2	2	3	3	3	4	4	4	...

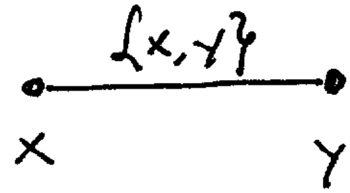
Graph Theory

Defn A *graph* is a pair of sets

$$G = (\underbrace{V}_{\text{vertex set}}, \underbrace{E}_{\text{edge set}})$$

where $V \neq \emptyset$ and $E \subseteq \underbrace{V^{(2)}}_{\text{2-element subsets of } V}$

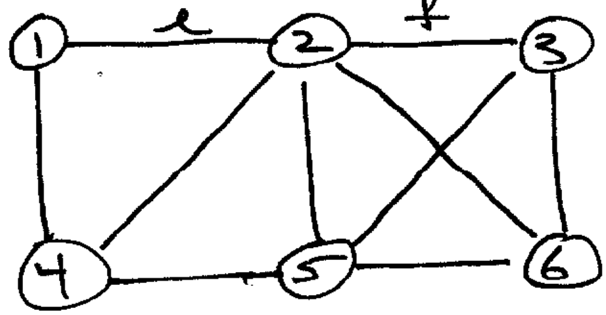
if $\{x, y\} \in E$:



notation: $\{x, y\} = xy = yx = \{y, x\}$

Ex.

G



$V(G) = \{1, 2, 3, 4, 5, 6\}$

$E(G) = \{12, 14, 23, 24, 25, 26, 35, 36, 45, 56\}$

we say:

- e <u>joins</u> 1 to 2
- e has <u>ends</u> 1 and 2
- e is <u>incident</u> to 1 (and 2)
- 1 is <u>adjacent</u> to 2
- e is <u>adjacent</u> to f

Let $x, y \in V(G)$. an <u>x-y Path</u> in G is a sequence

$$x = v_0, v_1, v_2, \ldots, v_k = y$$

of vertices satisfying

- $\{v_{i-1}, v_i\} \in E(G)$ for $1 \le i \le k$
- vertices are distinct, except possibly when $x = y$, in which case the Path is called <u>closed</u>.

The <u>length</u> of such a Path is k, the # of edges traversed. A Path of length 0 is called <u>trivial</u>. A non-trivial closed Path is called a <u>cycle</u>.

1 – 6 Path : 1, 4, 2, 3, 6 len = 4

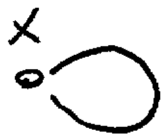
1 – 6 Path : 1, 2, 4, 5, 3, 6 len = 5

1 – 6 Path : 1, 2, 6 len = 2

4 – 4 Path : 4 len = 0, trivial

4 – 4 Path : 4, 2, 3, 6, 5, 4 len = 5
non-trivial
cycle.

<u>note</u>:



not allowed $\{x, x\} = \{x\}$

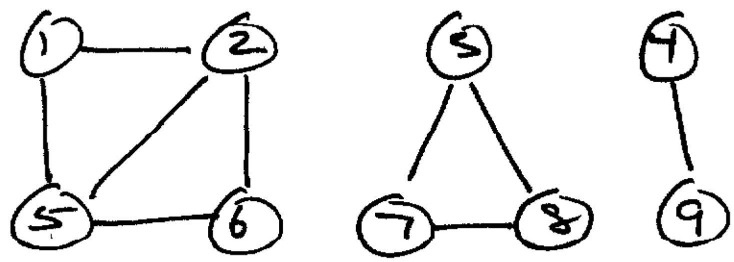
not allowed $\{x, y\} = \{x, y\}$

so smallest cycle length is 3:

Defn

A graph $G = (V, E)$ is called **connected** iff for all $x, y \in V$, G contains an x-y path.

Ex.



not connected

$$V = \{1, 2, 3, \ldots, 9\}$$

$$E = \{12, 25, 26, 15, 56, 37, 38, 78, 49\}$$

A graph that is not connected is called **disconnected**.

Defn

A <u>subgraph</u> H of a graph G is a graph with

$$V(H) \subseteq V(G)$$

and $E(H) \subseteq E(G)$.

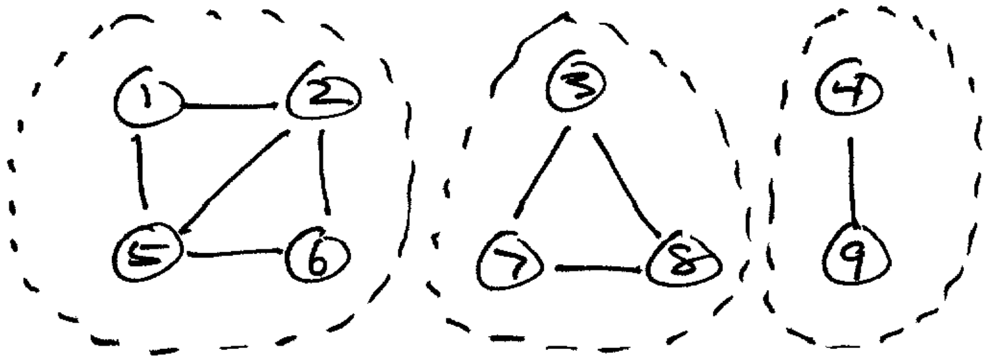
Defn

A <u>connected component</u> of a graph G is a subgraph H satisfying:

(1) H is connected

(2) H is maximal w.r.t. (1)

Ex.



conn. comps. of last ex.

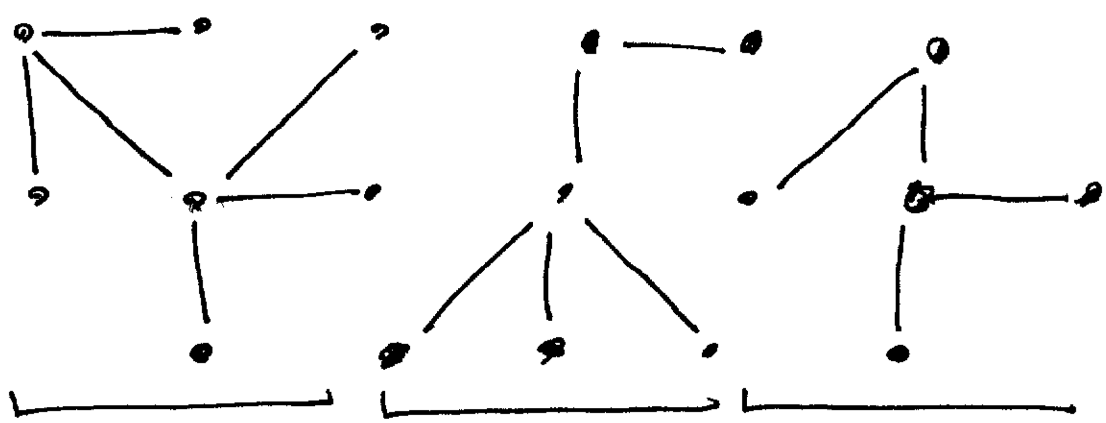
Defn

A Graph G is called *acyclic* (or a *forest*) iff it contains no cycles.

Defn

A *Tree* is a connected acyclic Graph.

Ex. *Acyclic*



	tree	tree	tree
#vert.	7	6	5
#edges.	6	5	4

<u>Theorem</u>

$$\forall n \geq 1 : \boxed{\text{if } T \text{ is a tree on } n \text{ vertices, then } T \text{ has } n-1 \text{ edges}} \leftarrow P(n)$$

<u>Proof:</u>

I. If $n=1$, then T has 1 vertex

$T \quad \overset{1}{\bullet} \qquad V=\{1\},\ E=\emptyset$.

and 0 edges. $\therefore$ the base case is satisfied.

IId. $\forall n>1 : P(1) \wedge \ldots \wedge P(n-1) \rightarrow P(n)$.

Let $n>1$ be arbitrary.

Assume for all k in the range $1 \le k < n$ that if T' is a tree on k vertices, then T' has $k-1$ edges. We must show that if T is a tree on n vertices, then T has $n-1$ edges.

Let T be a tree on n vertices. Let e be any edge of T, and remove it. This results in 2 trees T_1 and T_2, each with fewer than n vertices. Say T_1 has k_1 vertices, and T_2 has k_2 vertices. So $k_1 < n$ and $k_2 < n$. By the ind. hyp. T_1 has $k_1 - 1$ edges and T_2 has $k_2 - 1$ edges.

so the number of edges in T

is

$$(k_1 - 1) + (k_2 - \cancel{1}) + \cancel{1} = (k_1 + k_2) - 1$$

$$= n - 1,$$

since $k_1 + k_2 = n$ (no vertices were removed.) i.e. T has $n-1$ edges.

Result follows for all n by 2nd P.M.I. ∎