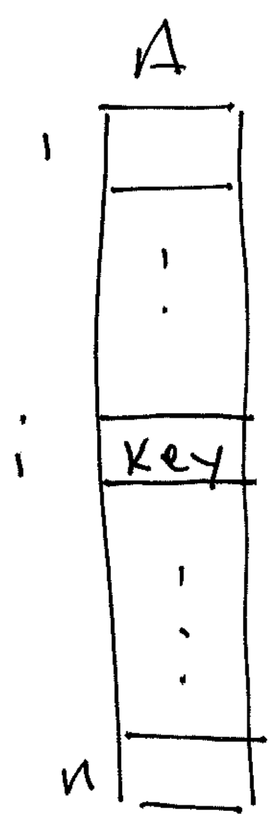
CSE 101 3-14-24

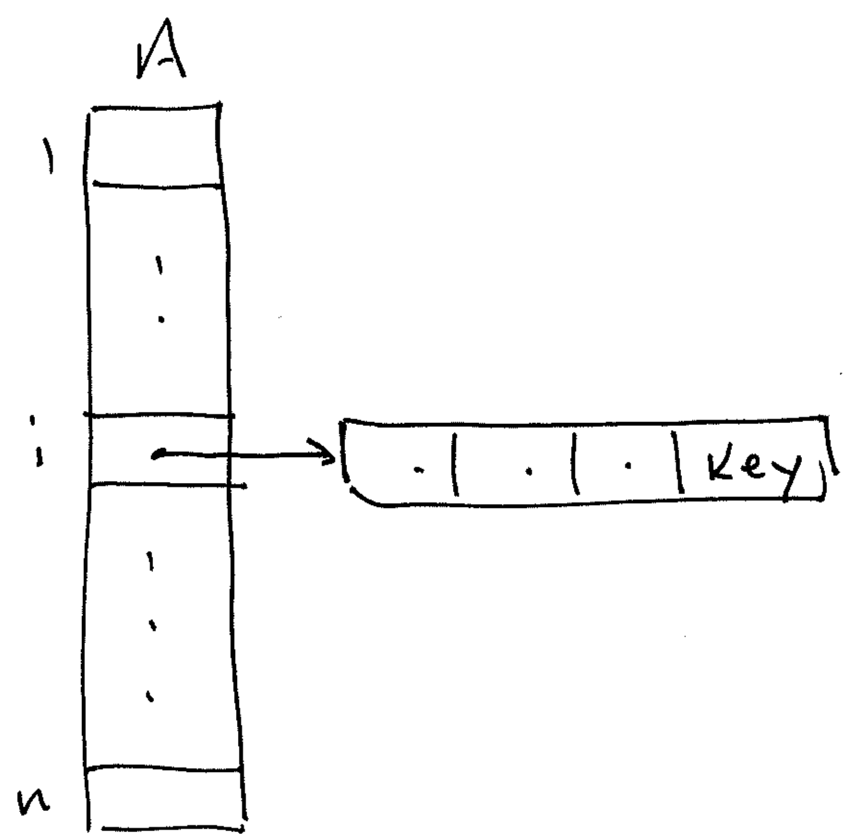
Heaps & Priority Queues

Our Picture



A[i] is a key

General Picture



A[i].key is a key

SSSP in Weighted Graphs

Defn

$G = (V, E, w)$ is a *weighted graph*

if $V = \{\text{vertex set}\}$, $E = \{\text{edge set}\}$,

$$w : E \to \mathbb{R}$$

let $P : x_0, x_1, \ldots, x_k$ be a path in G. the *weight* of P is

$$w(P) = \sum_{i=1}^{k} w(x_{i-1}, x_i)$$

$$P : \quad x_0 \bullet \to \circ \to \circ \quad \cdots \quad \circ \to \circ \; x_k$$

Shortest Path Weight

$$\delta(x,y)=\begin{cases}\min\{w(p) \mid p \text{ is an } x\text{-}y \text{ path}\} & \text{if } y \text{ is reachable from } x\\ \infty & \text{otherwise}\end{cases}$$

Shortest Path

any x-y path p s.t. $w(p)=\delta(x,y)$

SSSP Problem

Given source $s\in V(G)$: (1) determine $\delta(s,x)$ for all $x\in V(G)$, and (2) for all $x\in V(G)$ s.t. $\delta(s,x)<\infty$, find a shortest s-x path.

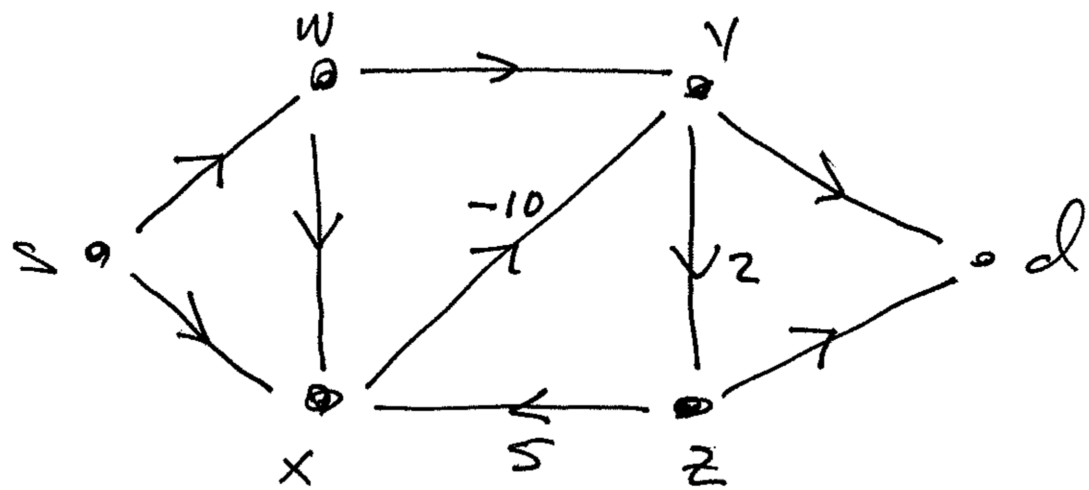
Two Solutions

- Bellman-Ford: + or - weights
  slower

- Dijkstra: + weights only
  faster

Both Algorithms actually find min-weight walks. This works since a min weight walk <u>is</u> a path.

Ex.



note:

$$w(x, y, z, x) = -3$$

If there is a neg. weight cycle reachable from source, then there is no min-weight s-d walk.

common infrastructure

Vertex attributes:

$p[x]$: Parent, encode a shortest path tree.

$d[x]$: estimate of $\delta(s,x)$

Predecessor subgraph: (V_p, E_p)

$$V_p = \{x \in V(G) \mid p[x] \neq nil \text{ or } x = s\}$$

$$E_p = \{(p[x], x) \mid p[x] \neq nil\}$$

Use PrintPath(G, s, x) to extract shortest paths.

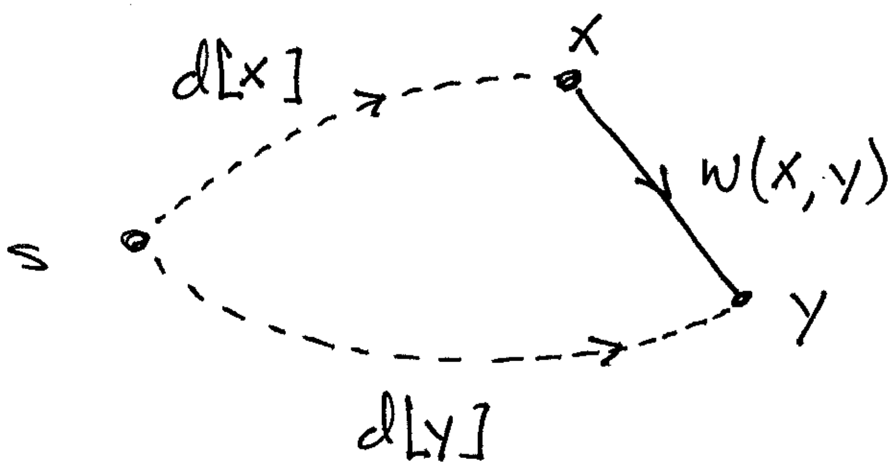
Helper functions:

- Initialize (G, s)

- Relax (x, y)

↓

Picture



Relax(x,y) changes y. After Relax(x,y), $d[y] \le d[x] + w(x,y)$ is true.

Lemma 1

Let $x \in V(G)$. Suppose after Initialize(G, s) some sequence of calls to Relax$(\cdot,\cdot)$ results in $d[x]$ being finite. Then G contains an s-x path of weight $d[x]$.

Lemma 2

After Initialize(G, s), the inequality

$$\delta(s,x) \leq d[x] \qquad (\forall x \in V(G))$$

is maintained over any sequence of calls to Relax$(\cdot,\cdot)$

Lemma 3 (Path Relaxation Property)

If $P: s = x_0, x_1, x_2, \ldots, x_k$ is a shortest s-x_k path, and the edges of P are relaxed in order

$$(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{k-1}, x_k)$$

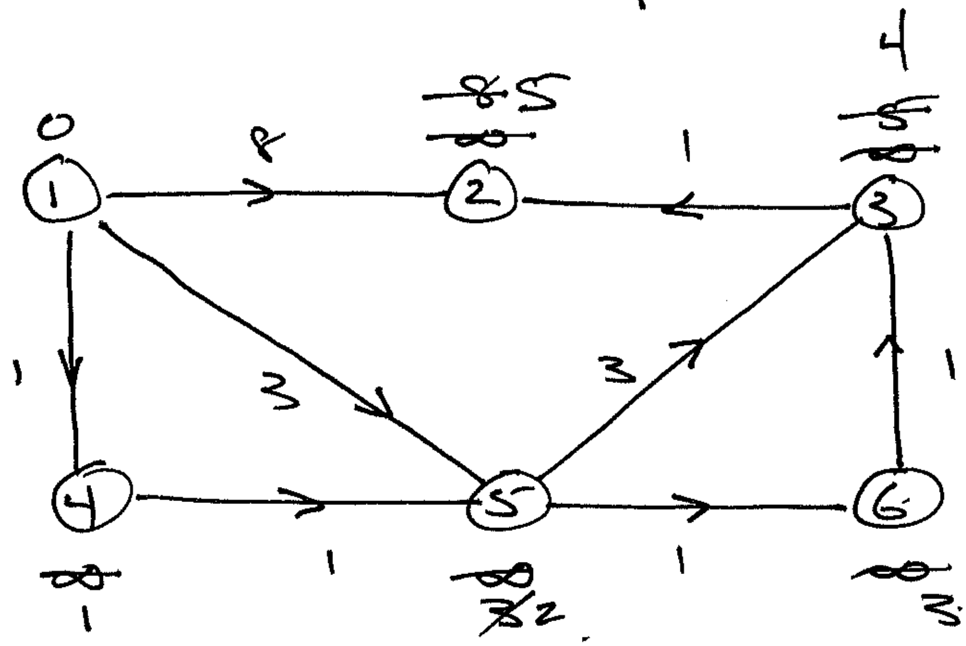
then $d[x_k] = \delta(s, x_k)$. This is true regardless of any other relaxations that occur, even if mixed with above.

Bellman-Ford:

$$\text{total cost} = \Theta(mn)$$

where $m = |E(G)|$ and $n = |V(G)|$

Ex. source = 1



extract order

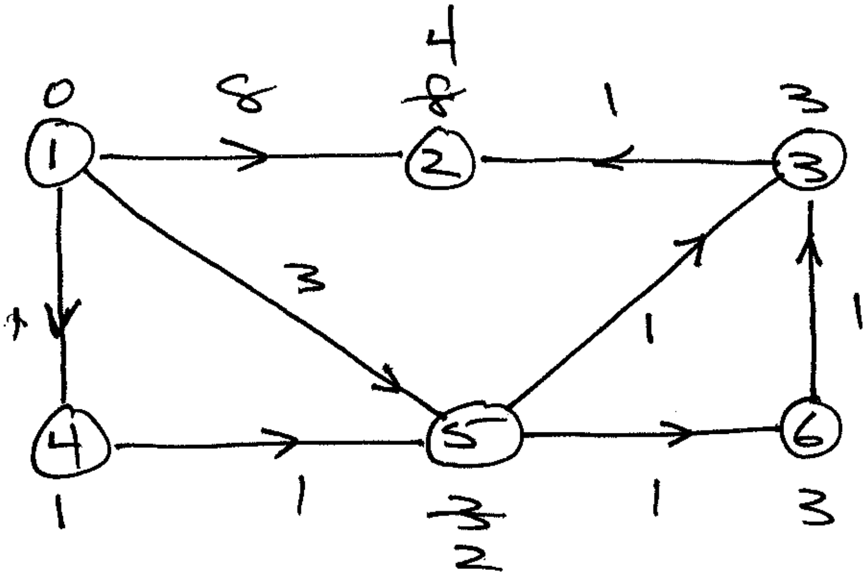
1
4
5
6
3
2

PQ : ~~1~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~ ~~6~~

d : 0 ~~8~~5 ~~5~~4 1 ~~3~~2 3

p : n ~~1~~3 ~~5~~6 1 ~~1~~4 5

Ex. source = 1



extract order

1
4
5
3
6
2

PQ :	~~1~~	~~2~~	~~3~~	~~4~~	~~5~~	~~6~~
d :	0	~~8~~4	3	1	~~3~~2	3
p :	n	~~1~~3	5	1	~~1~~4	5

Tree :