Defn

Let $f(n), g(n)$ be positive fcns. We write $f(n) = O(g(n))$ iff there exists $B > 0$, $n_0 > 0$ s.t.

$$\frac{f(n)}{g(n)} \le B$$

for all $n \ge n_0$.

We say: $g(n)$ is an <u>asymptotic upper bound</u> for $f(n)$.

Ex. $f(n) = 2n+5$, $g(n) = n$.

claim: $2n+5 = O(n)$ ✓

check:

$$\frac{2n+5}{n} \le \underset{B}{3} \quad \forall\, n \ge \underset{n_0}{5} \quad ✓$$

i.e. $2n+5 \le 3n \quad \forall\, n \ge 5$ ✓

i.e. $5 \le n \quad \forall\, n \ge 5$ ✓

Exercise:

show $an+b = O(n)$ for any $a > 0$ and b.

Defn

Write $f(n) = \Omega(g(n))$ iff

there exist $B > 0$, $n_0 > 0$ s.t.

$$B \leq \frac{f(n)}{g(n)}$$

for all $n \geq n_0$. We say

$g(n)$ is asymptotic lower bound

for $f(n)$.

Ex. $f(n) = 6n^3 + 4n$, $g(n) = 2n^2$

claim: $6n^3 + 4n = \Omega(2n^2)$

need

$$7 \leq \frac{6n^3+4n}{2n^2}$$

for all $n \geq 2$

i.e. $14n^2 \leq 6n^3 + 4n \quad \forall\, n \geq 2$

i.e. $7n \leq 3n^2 + 2 \quad \forall\, n \geq 2$

i.e. $3n^2 - 7n + 2 \geq 0 \quad \forall\, n \geq 2$

$\vdots$

<u>Exercise</u> show

$$an^3 + bn^2 + cn + d = \Omega(en^2 + fn + g)$$

for any $a>0$, $e>0$, b, c, d, f, g

Note:

$$\frac{f(n)}{g(n)} \le B \text{ iff } \frac{g(n)}{f(n)} \ge \frac{1}{B}$$

hence

$$f(n) = O(g(n)) \text{ iff } g(n) = \Omega(f(n))$$

Defn

write $f(n) = \Theta(g(n))$ iff there exist $B_1 > 0$, $B_2 > 0$, $n_0 > 0$ s.t.

$$B_1 \le \frac{f(n)}{g(n)} \le B_2$$

for all $n \ge n_0$.

Equivalently:

$f(n) = \Theta(g(n))$

iff both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

Ex. $f(n) = 5n^2 + 11n - 24$, $g(n) = n^2$

claim: $5n^2 + 11n - 24 = \Theta(n^2)$

need: $$4 \le \frac{5n^2 + 11n - 24}{n^2} \le 6$$

for all $n \ge 8$

check this

note:

we have

$$\begin{tikzcd}
f(n)=\Theta(g(n)) \arrow[d, leftrightarrow] & \text{iff} & g(n)=\Theta(f(n)) \arrow[d, leftrightarrow] \\
\begin{array}{c} f(n)=O(g(n)) \\ \text{and} \\ f(n)=\Omega(g(n)) \end{array} & \leftrightarrow & \begin{array}{c} g(n)=O(f(n)) \\ \text{and} \\ g(n)=\Omega(f(n)) \end{array}
\end{tikzcd}$$

Defn

write $f(n)=o(g(n))$ iff

$$\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right)=0.$$

We say: $g(n)$ is a strict asymptotic upper bound for $f(n)$.

Ex. $f(n) = \ln(n)$, $g(n) = n$.

claim: $\ln(n) = o(n)$ ✓

$$\lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right) = \lim_{n\to\infty}\left(\frac{1/n}{1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

Exercise: (a) $\log_b(n) = o(n)$ for any $b > 1$

(b) $\ln(n) = o(n^\alpha)$ for $\alpha > 0$

(c) $\log_b(n) = o(n^\alpha)$ for any $\alpha > 0$, $b > 1$.

Ex. $f(n) = n^k$ ($k \geq 0$ ~~integer~~), $g(n) = e^n$

claim: $n^k = o(e^n)$ ✓

i.e. $$\lim_{n\to\infty}\left(\frac{n^k}{e^n}\right) = \lim_{n\to\infty}\left(\frac{k\,n^{k-1}}{e^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{k(k-1)\,n^{k-2}}{e^n}\right)$$

$$\vdots$$

$$= 0$$ ✓

Exercise replace e by $b > 1$.

Defn

$f(n) = \omega(g(n))$ iff

$$\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = \infty$$

note :

$$\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = 0 \quad \text{iff} \quad \lim_{n \to \infty} \left(\frac{g(n)}{f(n)} \right) = \infty$$

hence

$f(n) = o(g(n))$ iff $g(n) = \omega(f(n))$

Analogy

is analogous to

$$f(n) = O(g(n)) \longleftrightarrow x \leq y$$

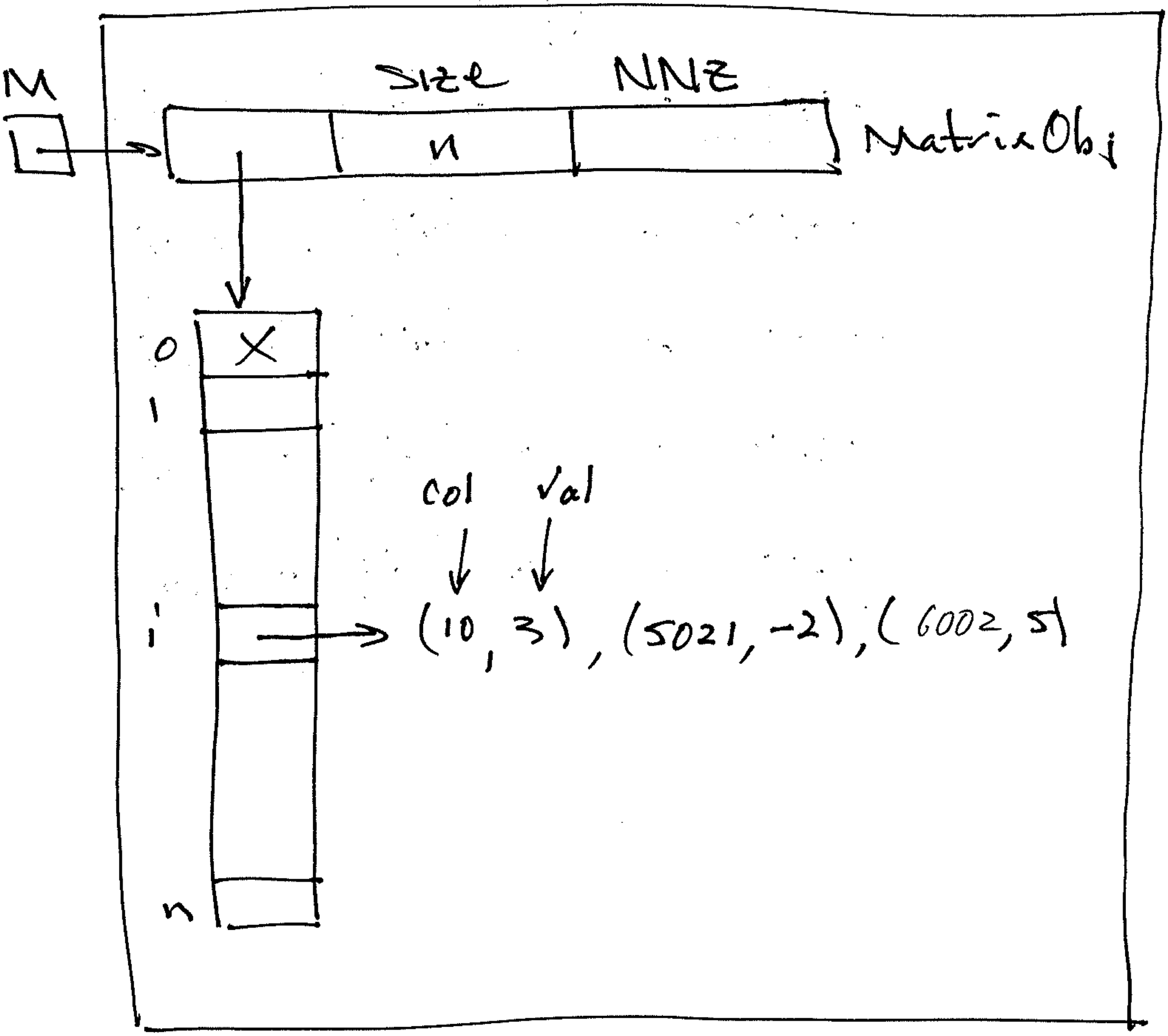
$$f(n) = \Omega(g(n)) \longleftrightarrow x \geq y$$

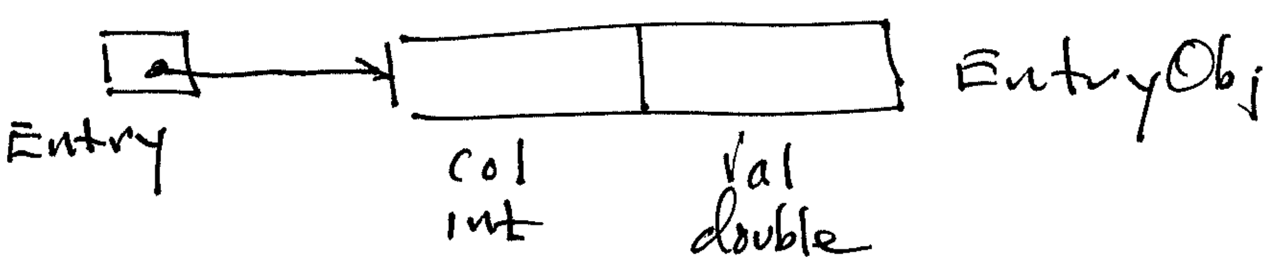
$$f(n) = \Theta(g(n)) \longleftrightarrow x = y$$

$$f(n) = o(g(n)) \longleftrightarrow x < y$$

$$f(n) = \omega(g(n)) \longleftrightarrow x > y$$

Pa4: Sparse matrices.





client view

$$\begin{array}{cc} & \begin{array}{cccccccccccccc} & & & 10 & & & & 5021 & & & & 6002 & & \end{array} \\ i & \left(\begin{array}{cccccccccccccc} 0 & \cdots & 0 & 3 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 & 5 & 0 & \cdots \end{array}\right) \end{array}$$