cse 101-01 2-7-23

Runtime of Algorithms

Defn

let $f(n)$, $g(n)$ be positive fcns. We write

$$f(n) = O(g(n))$$

to mean

there exist $B>0$, $n_0>0$ s.t.

$$\frac{f(n)}{g(n)} \leq B$$

for all $n \geq n_0$. We say "$g(n)$ is an asymptotic upper bound for $f(n)$".

Ex.

let $f(n) = 2n+5$, $g(n) = n$.

claim: $f(n) = O(g(n))$

$$2n+5 = O(n)$$

why? let $B = 6$, $n_0 = 2$

check that: if $n \geq 2$ then

$$\frac{2n+5}{n} \leq 6$$

i.e. $2 + \frac{5}{n} \leq 6$

or: $2n+5 \leq 6n$

$$5 \leq 4n \quad ✓$$

Exercise: show that

$$an + b = O(n)$$

for any $a > 0$, b.

<u>Defn</u>

We write $f(n) = \Omega(g(n))$ iff there exist $B > 0$, $n_0 > 0$ s.t. for all $n \geq n_0$:

$$B \leq \frac{f(n)}{g(n)}$$

We say: "$g(n)$ is an asymptotic lower bound for $f(n)$".

<u>Ex.</u> $f(n) = 6n^3 + 4n$, $g(n) = 2n^2$. check:

$$\frac{6n^3+4n}{2n^2} \geq \underset{B}{\underset{\uparrow}{7}}$$

for all $n \geq \underset{n_0}{\underset{\uparrow}{2}}$.

check:

$$6n^3 + 4n \geq 14n^2 \qquad \forall\, n \geq 2$$

i.e. $$6n^3 - 14n^2 + 4n \geq 0 \qquad \forall\, n \geq 2$$

i.e. $$3n^2 - 7n + 2 \geq 0 \qquad \forall\, n \geq 2$$

$\therefore$ $$6n^3 + 4n = \Omega(2n^2)$$

Exercise show

$$an^3 + bn^2 + cn + d = \Omega(en^2)$$

for any $a > 0$, $e > 0$, b, c, d.

Note:

$$\frac{f(n)}{g(n)} \le B \text{ iff } \frac{g(n)}{f(n)} \ge \frac{1}{B}$$

Hence:

$$f(n) = O(g(n)) \text{ iff } g(n) = \Omega(f(n))$$

Defn

write $f(n) = \Theta(g(n))$ iff both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

we say: $g(n)$ is a tight asymptotic bound for $f(n)$. Equivalently $\forall n \ge n_0$

$$B_1 \le \frac{f(n)}{g(n)} \le B_2$$

for some $B_1 > 0$, $B_2 > 0$, $n_0 > 0$

Ex $f(n) = 5n^2 + 11n - 24$, $g(n) = n^2$

claim: $5n^2+11n-24 = \Theta(n^2)$

check:

$$4 \leq \frac{5n^2+11n-24}{n^2} \leq 6$$

for all $n \geq 8$.

Defn

write $f(n) = o(g(n))$ iff

$$\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = 0$$

we say is a strict asymptotic upper bound for $f(n)$.

Ex. Let $f(n) = \ln(n)$, $g(n) = n$

$$\lim_{n \to \infty} \left(\frac{\ln n}{n} \right) = \lim_{n \to \infty} \left(\frac{1/n}{1} \right)$$

$$= \lim_{n \to \infty} \left(\frac{1}{n} \right) = 0$$

$$\therefore \quad \ln(n) = o(n)$$

Exercise for any $\alpha > 0$:

$$\ln(n) = o(n^\alpha)$$

Exercise

$$\log_b(n) = o(n^\alpha)$$

for any $b > 1$, $\alpha > 0$.

Ex.

$f(n) = n^k$ ($k \geq 0$ integer), $g(n) = e^n$

Then $n^k = o(e^n)$.

$$\lim_{n\to\infty}\left(\frac{n^k}{e^n}\right) = \lim_{n\to\infty}\left(\frac{k n^{k-1}}{e^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{k(k-1)n^{k-2}}{e^n}\right)$$

$$\vdots$$

$$= 0$$

Defn

write $f(n) = \omega(g(n))$ iff

$$\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = \infty$$

We say is a strict asymptotic lower bound for $f(n)$

note:

$$\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right)=0 \quad \text{iff} \quad \lim_{n\to\infty}\left(\frac{g(n)}{f(n)}\right)=\infty$$

hence

$$f(n)=o(g(n)) \quad \text{iff} \quad g(n)=\omega(f(n))$$

Analogy:

$$\begin{array}{lcl}
 & \text{analogous to} & \\
f(n)=O(g(n)) & \longleftrightarrow & x \le y \\
f(n)=\Omega(g(n)) & \longleftrightarrow & x \ge y \\
f(n)=\Theta(g(n)) & \longleftrightarrow & x = y \\
f(n)=o(g(n)) & \longleftrightarrow & x < y \\
f(n)=\omega(g(n)) & \longleftrightarrow & x > y
\end{array}$$

Part: Matrix ADT