CSE 101 2-3-22

Pa4:

helper fcn for Sum(A,B)

List addList(List P, List Q) {
moveFront on P and Q
while(on both P & Q)
walk down both ~~lists~~ simul.
advance on the one behind
write the behind one
into S. If elts.
are same add and append
write into S, adv. on both.
}
while (still on P) {
copy P into S
}
while (still on Q) {
copy Q into S
}
return S
}

- Part ext. 1 more day

Back to Asymptotic Growth of functions.

So far

- $O(g(n))$: $\frac{f(n)}{g(n)} \leq B \quad \forall n \geq n_0$
- $\Omega(g(n))$: $B \leq \frac{f(n)}{g(n)} \quad \forall n \geq n_0$
- $\Theta(g(n))$: $B_1 \leq \frac{f(n)}{g(n)} \leq B_2 \quad \forall n \geq n_0$

Defn.
$f(n) = o(g(n))$ iff $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)}\right) = 0$

we say: • $g(n)$ is a strict asym. U.b. $f(n)$

• $f(n)$ is strictly asym. bdd above by $g(n)$.

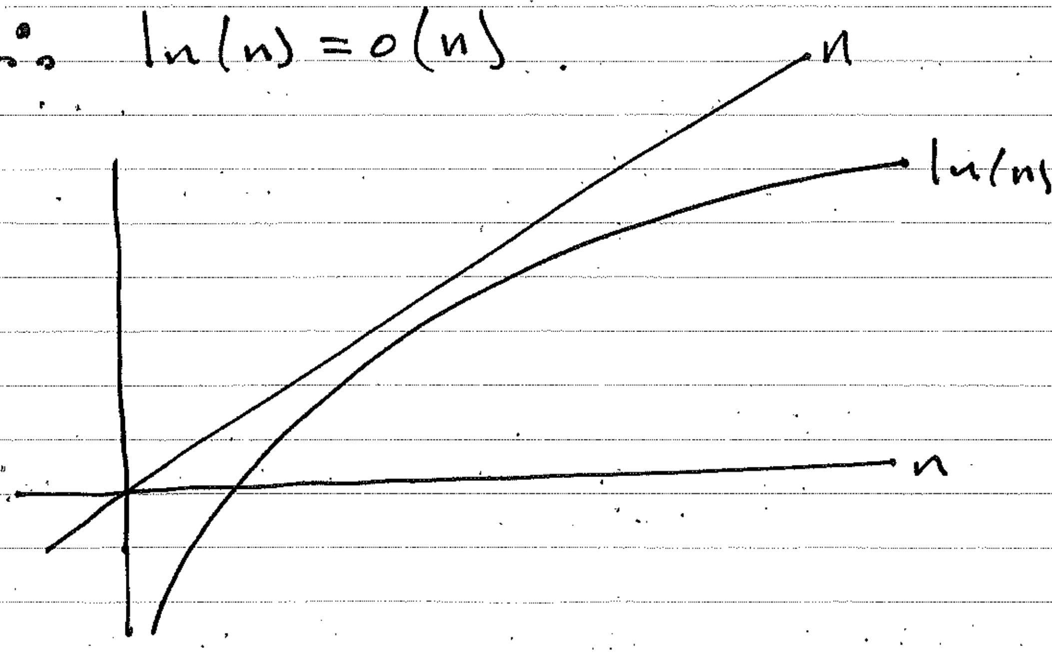
means: growth rate of $f(n)$ is strictly slower than that of $g(n)$

Ex. $f(n) = \ln(n)$, $g(n) = n$

$$\lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right) = \lim_{n\to\infty}\left(\frac{1/n}{1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

$\therefore$ $\ln(n) = o(n)$



Ex. $(\ln(n))^2 = o(n)$

$$\lim_{n\to\infty}\left(\frac{\ln(n)^2}{n}\right) = \lim_{n\to\infty}\left(\frac{2\ln(n)\cdot\frac{1}{n}}{1}\right)$$

$$= 2\cdot\lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right) = 0$$

<u>Exercise</u>

show $(\ln(n))^k = o(n) \quad \forall\ k \geq 0$, $k \in \mathbb{R}$.

<u>Exercise</u>

show for any $b > 1$, and $k \geq 0$

$$(\log_b(n))^k = o(n)$$

<u>Hint:</u> $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$.

<u>Ex.</u> $f(n) = n^k \quad (k \geq 0 \ , \ k \in \mathbb{R})$

$g(n) = e^n$

Then $f(n) = o(g(n))$ ✓

$$\left.\begin{aligned}\lim_{n\to\infty}\left(\frac{n^k}{e^n}\right) &= \lim_{n\to\infty}\left(\frac{k n^{k-1}}{e^n}\right)\\ &= \lim_{n\to\infty}\left(\frac{k(k-1)n^{k-2}}{e^n}\right)\\ &\ \ \vdots\\ &= \lim_{n\to\infty}\left(\frac{\text{const.}}{\text{blows up}}\right) = 0\end{aligned}\right] \begin{aligned}&\#\text{ of apps. of L'hop.}\\ &= \lceil k \rceil\end{aligned}$$

Exercise

let $b > 1$, $k \geq 0$, $k \in \mathbb{R}$.

show $n^k = o(b^n)$.

Ex. let $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$

Then $n^\alpha = o(n^\beta)$.

$$\lim_{n \to \infty} \left(\frac{n^\alpha}{n^\beta} \right) = \lim_{n \to \infty} n^{(\alpha - \beta)} = 0$$

Remark:

{all log fcns} << {all polys} << {all exps}

little o of

<u>Defn</u>

$f(n) = \omega(g(n))$ iff $\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = \infty$

we say:

- g is a strict asym. lower bd.
- f is strictly asym. bdd below by g.

means: f(n) grows strictly faster than g(n).

<u>Remark</u>:

since $\lim_{n\to\infty} \left(\frac{1}{h(n)}\right) = \frac{1}{\lim_{n\to\infty} (h(n))}$,

we have

$$\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = \infty \text{ iff } \lim_{n\to\infty} \left(\frac{g(n)}{f(n)}\right) = 0$$

$\therefore$ $f(n) = \omega(g(n))$ iff $g(n) = o(f(n))$.

Ex.

Show if $1 < a < b$ then

$$a^n = o(b^n) \checkmark \qquad \frac{a}{b} < 1$$

and hence

$$b^n = \omega(a^n) \checkmark$$

$$\lim_{n\to\infty} \left(\frac{a^n}{b^n}\right) = \lim_{n\to\infty} \left(\frac{a}{b}\right)^n = 0.$$

Analogy with numbers:

analogous to

$$f(n) = O(g(n)) \longleftrightarrow x \le y$$

$$f(n) = \Omega(g(n)) \longleftrightarrow x \ge y$$

$$f(n) = \Theta(g(n)) \longleftrightarrow x = y$$

$$f(n) = o(g(n)) \longleftrightarrow x < y$$

$$f(n) = \omega(g(n)) \longleftrightarrow x > y$$

Exercises

(1) $\forall\, \alpha, \beta \in \mathbb{R}$: $$n^\alpha = \begin{cases} o(n^\beta) & \text{if } \alpha < \beta \\ \Theta(n^\beta) & \alpha = \beta \\ \omega(n^\beta) & \alpha > \beta \end{cases}$$

(2) $\forall\, a, b \in \mathbb{R}$, $a > 1$, $b > 1$

$$a^n = \begin{cases} o(b^n) & a < b \\ \Theta(b^n) & a = b \\ \omega(b^n) & a > b \end{cases}$$

(3) for any fcn $f(n)$, $c > 0$

$$\begin{aligned} c \cdot f(n) &= O(f(n)) \\ \text{''} \quad &= \Omega(f(n)) \\ \text{''} \quad &= \Theta(f(n)) \\ \text{''} \quad &= o(f(n)) \\ \text{''} \quad &= \omega(f(n)) \end{aligned}$$

show $c \cdot f(n) = O(f(n))$

must show $\exists B > 0, \exists n_0 > 0$
s.t. $\forall n \geq n_0$

$$\frac{c\,\cancel{f(n)}}{\cancel{f(n)}} \leq B$$

i.e. $c \leq B$

Pick $B = c+1$

(5) let $L = \lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right)$ if it exists.

(a) $0 \leq L < \infty \Rightarrow f(n) = O(g(n))$ $(\not\Leftarrow)$
(b) $0 < L \leq \infty \Rightarrow f(n) = \Omega(g(n))$ $(\not\Leftarrow)$
(c) $0 < L < \infty \Rightarrow f(n) = \Theta(g(n))$ $(\not\Leftarrow)$
(d) $L = 0 \Leftrightarrow f(n) = o(g(n))$
(e) $L = \infty \Leftrightarrow f(n) = \omega(g(n))$

(6) Rank the following fcns from lowest to highest asym. growth rate

n^2, $\ln(n)$, $(\ln(n))^2$, $n\ln(n)$

$\sqrt{n}$, $n\sqrt{n}$, $\ln(\ln(\sqrt{n}))$

$2^{\ln(n)}$, 2^n, 2^{3^n}, 3^{2^n}

<u>Ex.</u>

<u>SelectionSort(A)</u>

```
n = len[A]
for i = 1 to n-1
    imin = i
    for j = i+1 to n
        if A[j] < A[imin]
            imin = j
    A[i] <=> A[imin]      // swap
```

$$\underbrace{A_1 \cdots A_{i-1}}_{\text{sorted}} \; \underbrace{A_i \, A_{i+1} \cdots A_n}_{\text{unsorted}}$$

sorted: $(i-1)$ smallest elements in A

unsorted.

choose basic op. to be comparison of array elements (line 5)

Then

$i=1$, $2 \le j \le n$ # comp $= n-1$
$i=2$, $3 \le j \le n$ # comp $= n-2$
$\vdots$
$i=n-2$, $n-1 \le j \le n$ # comp $= 2$
$i=n-1$, $n \le j \le n$ # comp $= 1$

Thus

$$\#\text{comp} = (n-1)+(n-2)+\cdots+3+2+1$$
$$= \frac{n(n-1)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n = \Theta(n^2)$$

Next time: BFS, DFS.