CSE 101   2-2-21

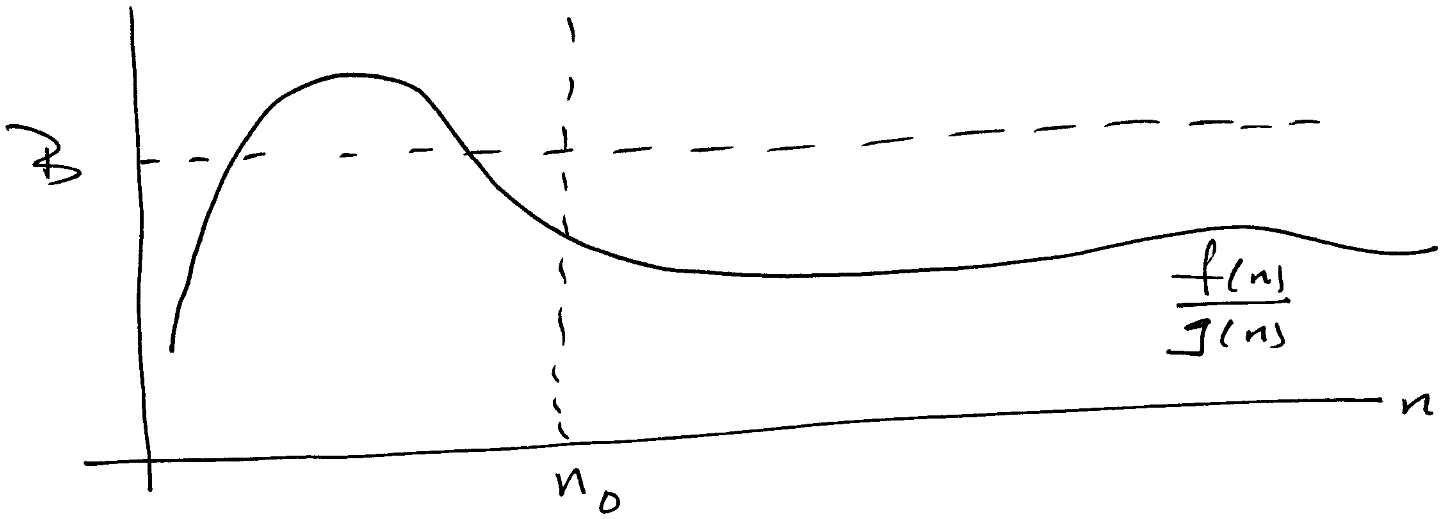
Asymptotic growth rates

Let $f(n)$ and $g(n)$ denote positive functions.

Defn 1

$f(n) = O(g(n))$ iff there exist constants $B > 0$, and $n_0 > 0$ such that

$$\frac{f(n)}{g(n)} \le B \quad \text{for all } n \ge n_0$$



We say $g(n)$ is an asymptotic upper bound for $f(n)$.

Ex

$f(n) = 2n+5$, $g(n) = n$.

$$\frac{f(n)}{g(n)} = \frac{2n+5}{n} = 2 + \frac{5}{n} \leq 3$$

B

for all $n \geq 5$. $\therefore$ $2n+5 = O(n)$

n_0

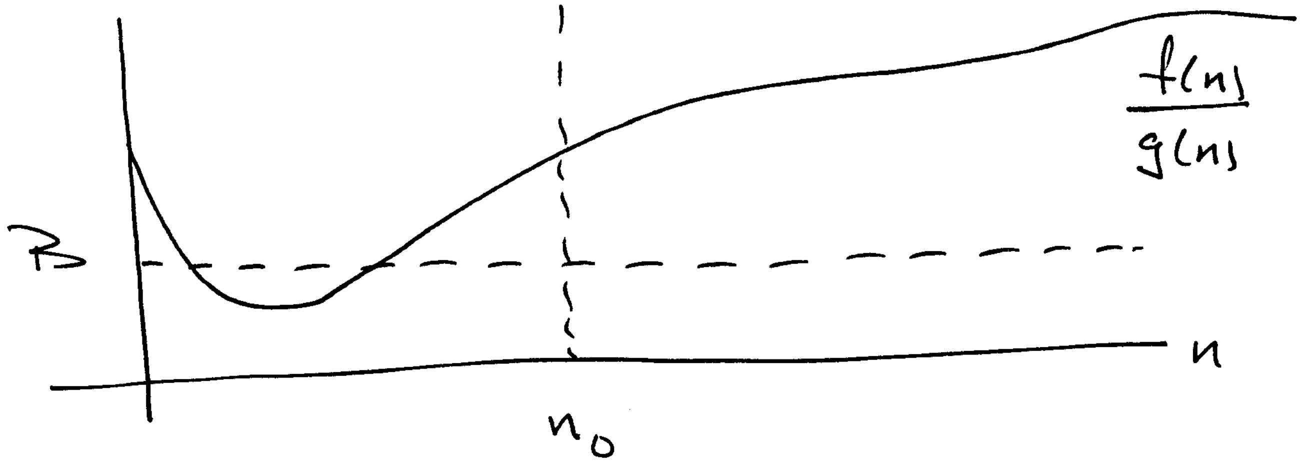
In fact $an+b = O(n)$ for any $a>0$, $b \in \mathbb{R}$. (Let $B = a+1$ and $n_0 = \lceil |b| \rceil$.)

Defn 2

$f(n) = \Omega(g(n))$ iff there exist constants $B>0$, $n_0>0$ s.t.

$$B \leq \frac{f(n)}{g(n)}$$

for all $n \geq n_0$. Say $g(n)$ is an asymp. L.B for $f(n)$.



Ex.

$f(n) = 6n^3 + 4n$, $g(n) = 2n^2$.

$$\frac{f(n)}{g(n)} = 3n + \frac{2}{n} \geq \overset{B}{7}$$

for all $n \geq \overset{n_0}{2}$ (check this.)

$\therefore \quad 6n^3 + 4n = \Omega(2n^2)$

- $f(n) = O(g(n))$ means: $f(n)$ grows no faster than $g(n)$.
- $f(n) = \Omega(g(n))$ means: $f(n)$ grows no slower than $g(n)$.

observe

$$f(n) = O(g(n)) \text{ iff } g(n) = \Omega(f(n))$$

since

$$\frac{f(n)}{g(n)} \le B_1 \quad \text{iff} \quad \frac{g(n)}{f(n)} \ge B_2$$

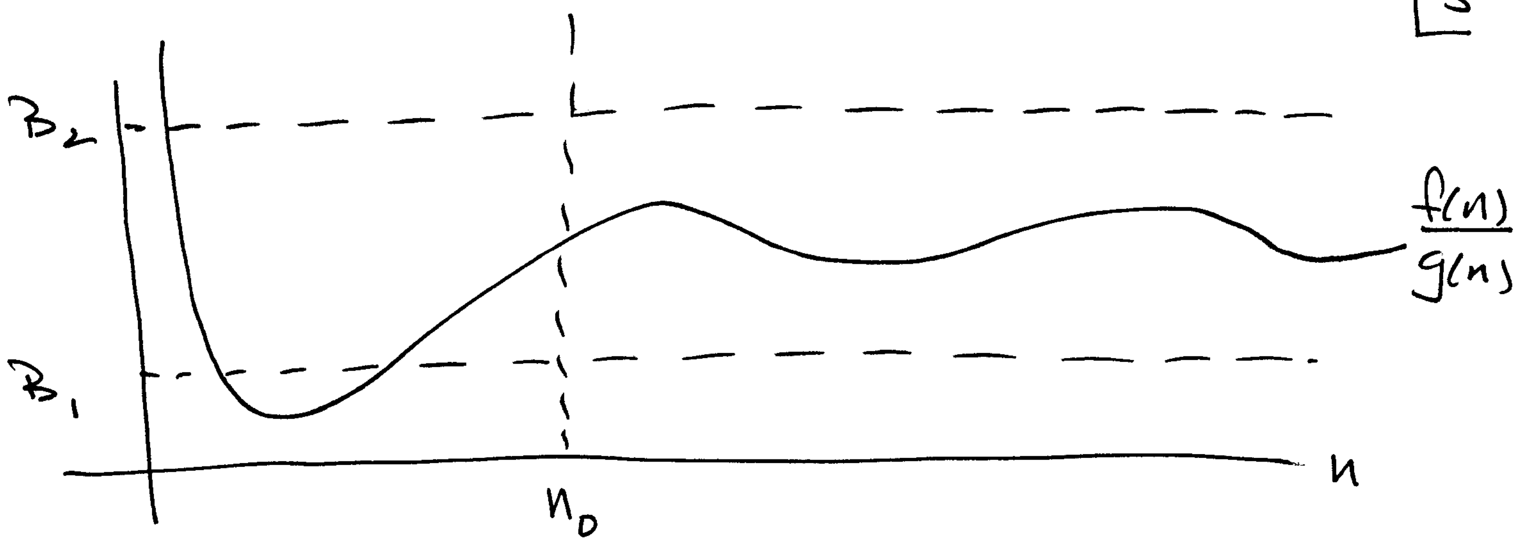
where $B_2 = \dfrac{1}{B_1}$.

<u>Defn 3</u>

$f(n) = \Theta(g(n))$ iff there exist constants B_1, B_2, n_0 s.t.

$$B_1 \le \frac{f(n)}{g(n)} \le B_2$$

for all $n \ge n_0$.



we say $g(n)$ is a tight asymptotic bound for $f(n)$.

<u>interpretation</u>: $f(n)$ grows at same rate as $g(n)$.

obviously

$f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

iff $g(n) = \Omega(f(n))$ and $g(n) = O(f(n))$

iff $g(n) = \Theta(f(n))$

Ex.

$f(n) = 5n^2 + 11n - 24$, $g(n) = n^2$. check that

$$\underbrace{4}_{B_1} \le \frac{5n^2+11n-24}{n^2} \le \underbrace{6}_{B_2}$$

for all $\underbrace{n \ge 8}_{n_0}$.

Exercise: for any $a, b, c \in \mathbb{R}$ with $a > 0$! $an^2 + bn + c = \Theta(n^2)$

In fact: if $f(n)$ is a polynomial of degree k, then

$$f(n) = \Theta(n^k).$$

Defn 4

$f(n) = o(g(n))$ iff $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = 0$

we say $g(n)$ is a <u>strict asymptotic upper bound</u> for $f(n)$.

<u>interpretation</u>: $f(n)$ grows strictly slower than $g(n)$.

<u>Ex</u>

$f(n) = \ln(n)$, $g(n) = n$.

$$\lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right) = \lim_{n\to\infty}\left(\frac{1/n}{1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

$\therefore \ln(n) = o(n)$

note: show that

$$\log_b(n) = o(n^k)$$

for any $b > 1$, and $k > 0$.

Ex.
$f(n) = n^k$ ($k \geq 0$ an integer), $g(n) = e^n$

$$\left.\begin{aligned} \lim_{n\to\infty}\left(\frac{n^k}{e^n}\right) &= \lim_{n\to\infty}\left(\frac{k n^{k-1}}{e^n}\right) \\ &= \lim_{n\to\infty}\left(\frac{k(k-1)n^{k-2}}{e^n}\right) \\ &\vdots \\ &= \lim_{n\to\infty}\left(\frac{k(k-1)\cdots 1}{e^n}\right) \end{aligned}\right\}$$

k applications of l'Hopital's rule.

$$= 0$$

$\therefore\ n^k = o(e^n)$

note: if $k \geq 0$ is not an integer, need $\lceil k \rceil$ applications of l'Hop.

note also: $n^k = o(b^n)$ for any

$k \geq 0$, $b > 1$.

observe:

$$\text{logs.} \ll \text{poly.} \ll \text{exp.}$$

"much slower growing"

Ex. $f(n) = n^\alpha$, $g(n) = n^\beta$ where $0 \leq \alpha < \beta$.

Then

$$\lim_{n\to\infty}\left(\frac{n^\alpha}{n^\beta}\right) = \lim_{n\to\infty}\left(n^{\alpha-\beta}\right)$$

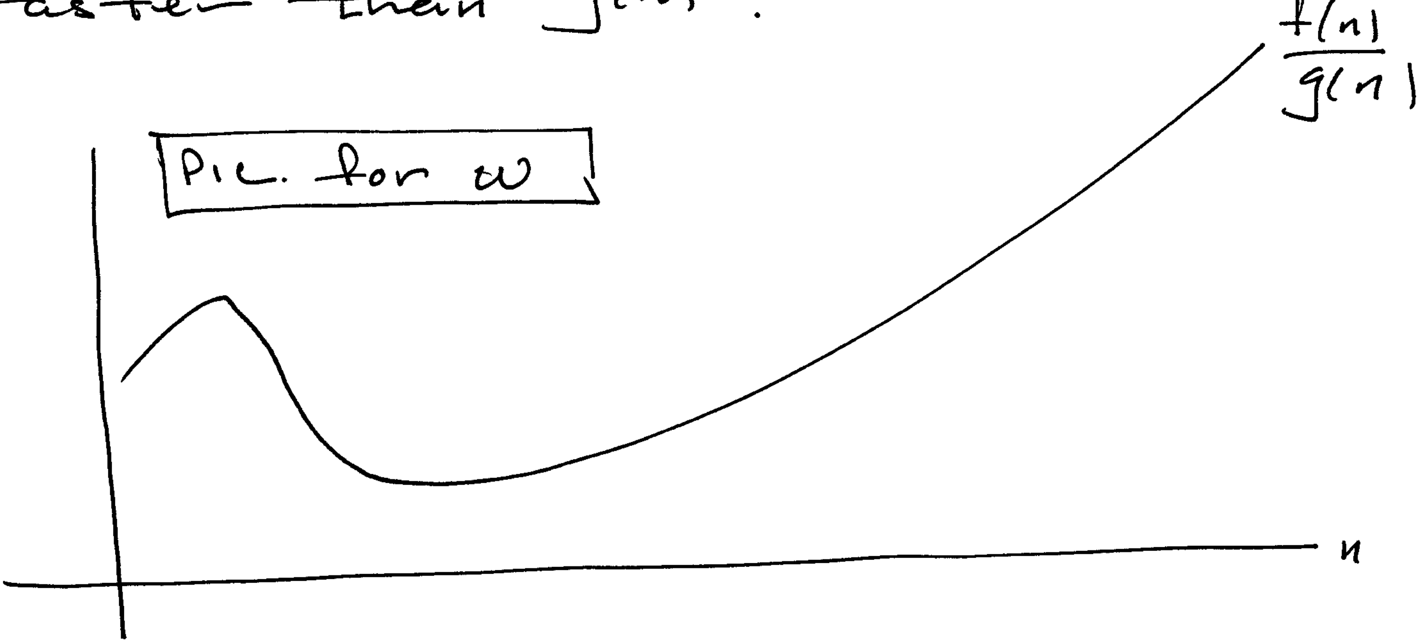
$$= \lim_{n\to\infty}\left(\frac{1}{n^{\beta-\alpha}}\right) = 0$$

$\therefore \quad n^\alpha = o(n^\beta)$

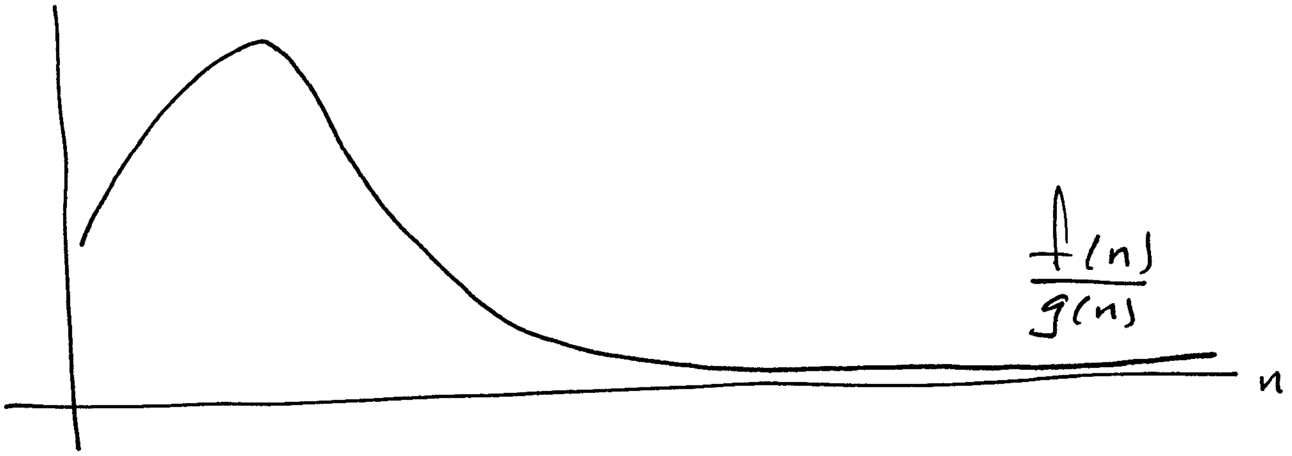
<u>Defn 5</u>

$f(n) = \omega(g(n))$ iff $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = \infty$. We say $g(n)$ is a strict asymptotic lower bound for $f(n)$.

<u>interpretation</u>: $f(n)$ grows strictly faster than $g(n)$.



Pic. for o



note

- reciprocal of a limit is limit of reciprocal.
  (convention: $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.)

- $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ iff $\lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$

- $f(n) = o(g(n))$ iff $g(n) = \omega(f(n))$

Preceding examples:

$n^k = \omega(\log_b(n))$ $\quad (k>0,\ b>1)$

$b^n = \omega(n^k)$ $\quad (k\geq 0,\ b>1)$

$n^\beta = \omega(n^\alpha)$ $\quad (0 \leq \alpha < \beta)$

Ex.

$f(n) = b^n$, $g(n) = a^n$ where $1 \le a < b$. Then

$$\lim_{n\to\infty}\left(\frac{b^n}{a^n}\right) = \lim_{n\to\infty}\left(\frac{b}{a}\right)^n = \infty \quad \left(\text{since } \frac{b}{a} > 1\right)$$

$\therefore \quad b^n = \omega(a^n)$

also: $a^n = o(b^n)$.

Analogy

$f(n) = O(g(n))$ analogous to $x \le y$

$f(n) = \Omega(g(n))$ $\longleftrightarrow$ $x \ge y$

$f(n) = \Theta(g(n))$ $\longleftrightarrow$ $x = y$

$f(n) = o(g(n))$ $\longleftrightarrow$ $x < y$

$f(n) = \omega(g(n))$ $\longleftrightarrow$ $x > y$

Transitive law:

$x \le y$ and $y \le z$, then $x \le z$

Similarly

✓ $f(n) = O(g(n))$ and $g(n) = O(h(n))$ implies $f(n) = O(h(n))$

why?

$$\frac{f(n)}{g(n)} \le B_1 \text{ and } \frac{g(n)}{h(n)} \le B_2 \text{ implies}$$

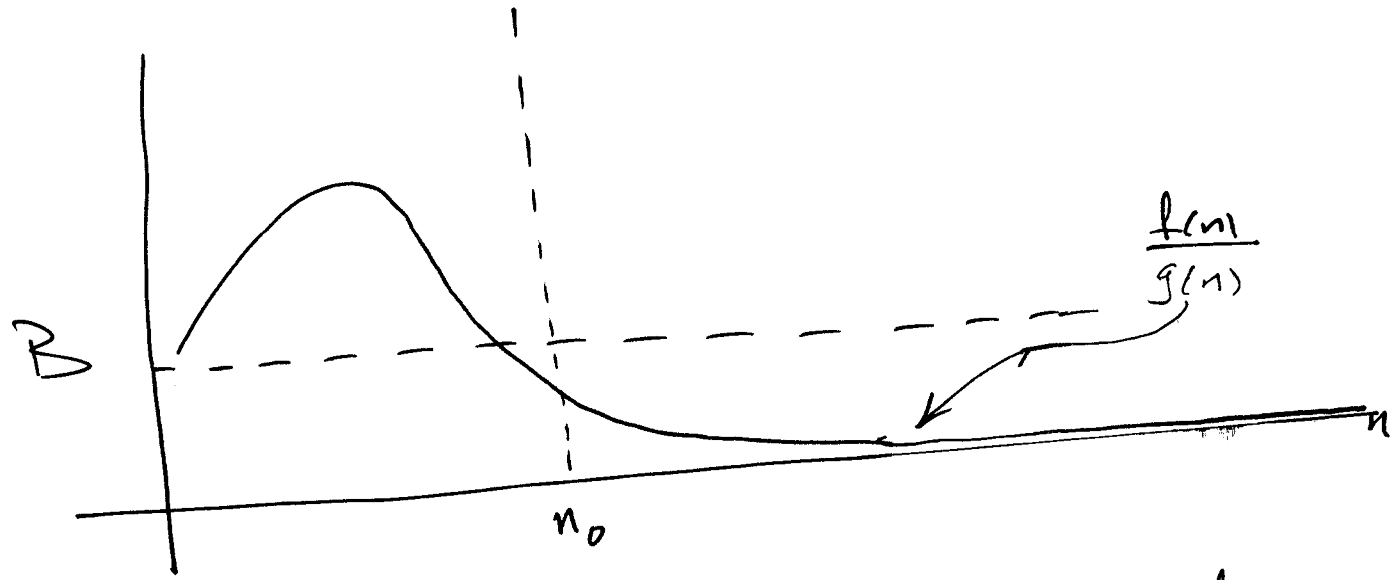
$$\frac{f(n)}{h(n)} = \frac{f(n)}{g(n)} \cdot \frac{g(n)}{h(n)} \le B_1 \cdot B_2$$

Exercise establish transitive law for Ω, Θ, o, ω.

Also: $x < y$ implies $x \le y$

Likewise: $f(n) = o(g(n))$ implies $f(n) = O(g(n))$

why? $\frac{f(n)}{g(n)} \to 0$



any $B > 0$ is an upper bound for $\frac{f(n)}{g(n)}$ for suff. large n.

$\therefore f(n) = O(g(n))$

Exercise: establish that

$f(n) = \omega(g(n))$ implies $f(n) = \Omega(g(n))$.

<u>Also</u> : $x < y$ implies $x \not> y$

<u>likewise</u>: $f(n) = o(g(n))$ implies $f(n) \neq \Omega(g(n))$

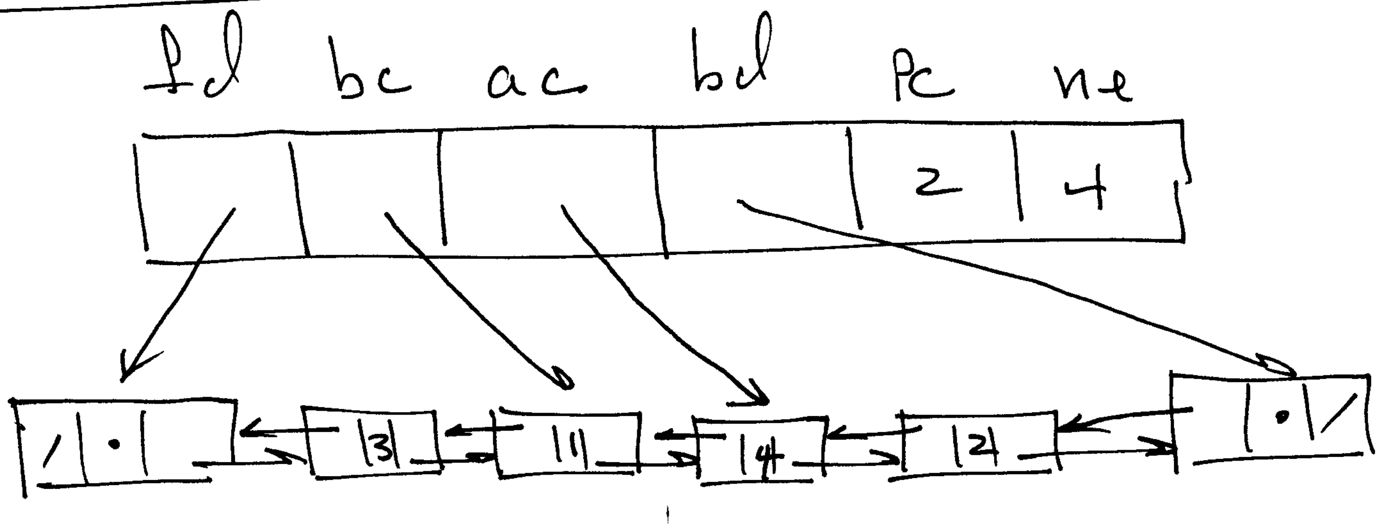
Exercise: establish this!

next time ....

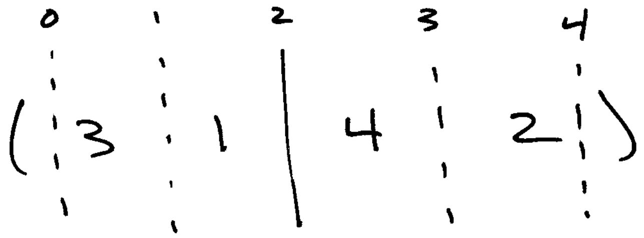
- heirarchy
- Insertion sort ?
- BFS, DFS

Pa4 stuff

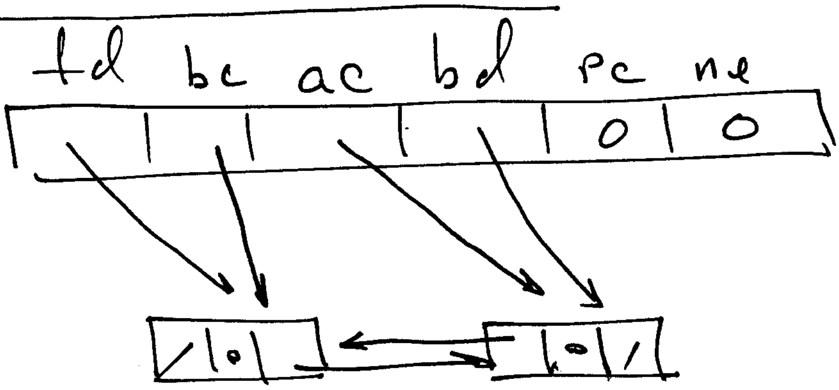
inside view



client view



empty state: inside:



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