CSE 101-02 5-2-23

Recall:

- $f(n) = O(g(n))$ iff there exist positive B, n_0 s.t. for all $n \geq n_0$:

$$\frac{f(n)}{g(n)} \leq B.$$

- $f(n) = \Omega(g(n))$ iff there exist pos. B, n_0 s.t. for all $n \geq n_0$

$$B \leq \frac{f(n)}{g(n)}.$$

- $f(n) = \Theta(g(n))$ iff there exist pos. B_1, B_2, n_0 s.t. for all $n \geq n_0$

$$B_1 \leq \frac{f(n)}{g(n)} \leq B_2$$

analogy:

$$f(n) = O(g(n)) \longleftrightarrow x \leq y$$

$$f(n) = \Omega(g(n)) \longleftrightarrow x \geq y$$

$$f(n) = \Theta(g(n)) \longleftrightarrow x = y$$

$$f(n) = o(g(n)) \longleftrightarrow x < y$$

$$f(n) = \omega(g(n)) \longleftrightarrow x > y$$

Defn

write $f(n) = o(g(n))$ iff

$$\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = 0$$

we say $g(n)$ is a strict asymptotic upper bound for $f(n)$.

Ex. $\ln(n) = o(n)$ ✓

$$\lim_{n \to \infty} \left(\frac{\ln(n)}{n} \right) = \lim_{n \to \infty} \left(\frac{\frac{1}{n}}{1} \right)$$

$$= \lim_{n \to \infty} \left(\frac{1}{n} \right) = 0$$

Also: $\ln(n) = o(n^k)$ (any $k>0$)

$$\lim_{n\to\infty}\left(\frac{\ln(n)}{n^k}\right) = \lim_{n\to\infty}\left(\frac{1/n}{kn^{k-1}}\right)$$

$$= \frac{1}{k}\lim_{n\to\infty}\left(\frac{1}{n^k}\right) = 0$$

Also: $\log_b(n) = o(n)$ (for any $b>1$)

why? $\log_b(n) = \dfrac{\ln(n)}{\ln(b)} = \left(\dfrac{1}{\ln b}\right)\cdot \ln n$

$$= \text{const} \cdot \ln n$$

$\therefore$

<u>Ex</u> if $0 \le \alpha < \beta$, then $n^\alpha = o(n^\beta)$

$$\lim_{n\to\infty}\left(\frac{n^\alpha}{n^\beta}\right) = \lim_{n\to\infty}(n^{\alpha-\beta}) = 0$$

Defn

write $f(n) = \omega(g(n))$ iff

$$\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = \infty$$

we say $g(n)$ is a strict asymptotic lower bound for $f(n)$.

Recall (from calculus)

Limit of a reciprocal is the recip. of the limit

i.e. $\lim \left(\frac{f(n)}{g(n)} \right) = 0$ iff $\lim_{n \to \infty} \left(\frac{g(n)}{f(n)} \right) = \infty$

i.e. $f(n) = o(g(n))$ iff $g(n) = \omega(f(n))$

Ex. $e^n = \omega(n^k)$ ✓ (for any $k>0$)

$$\lim_{n\to\infty}\left(\frac{e^n}{n^k}\right) = \lim_{n\to\infty}\left(\frac{e^n}{kn^{k-1}}\right)$$

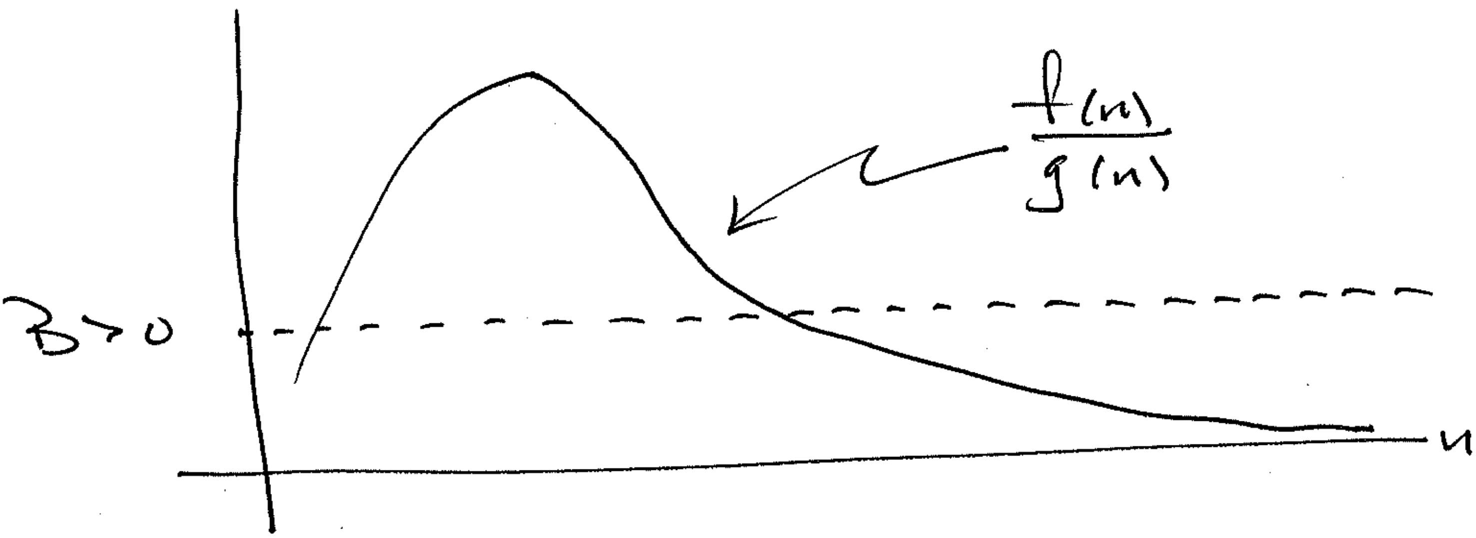
$$\vdots$$

$$= 0 \quad \left\{ \begin{array}{l} \text{after } \lceil k \rceil \text{ application} \\ \text{of l'Hop.} \end{array} \right.$$

Exercise if $b>1$, then for any $k>0$:

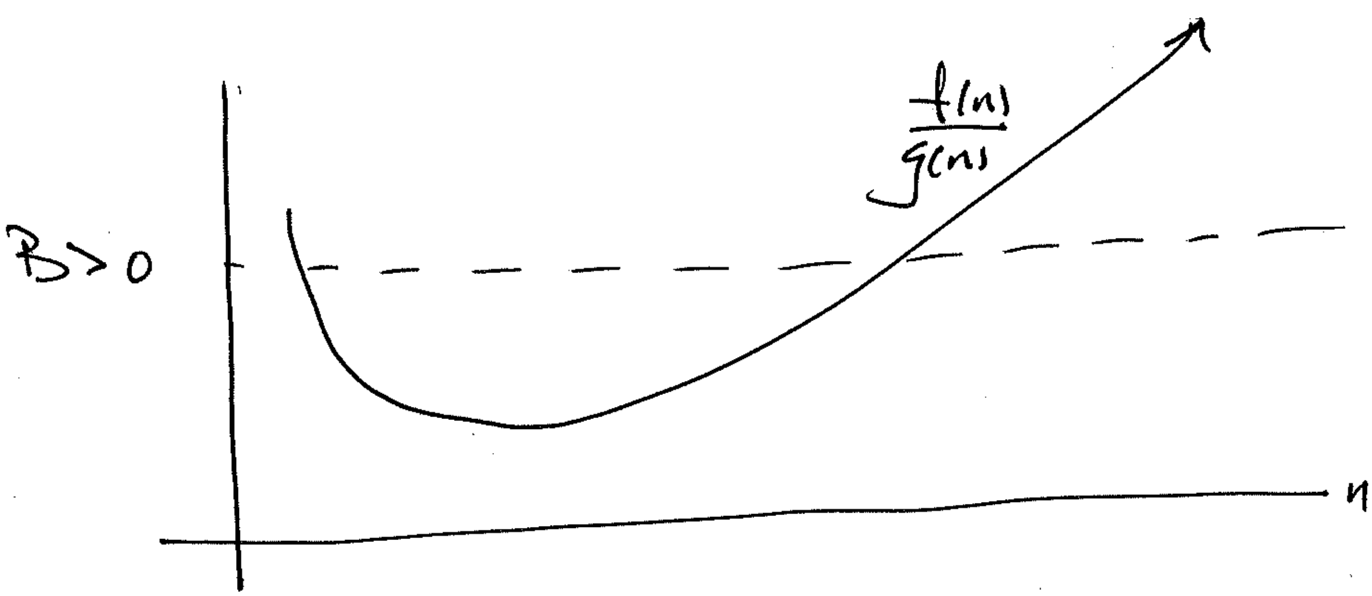
$$b^n = \omega(n^k)$$

Picture: $f(n) = o(g(n))$



<u>note</u>: $f(n) = o(g(n)) \Longrightarrow f(n) = O(g(n))$

Picture: $f(n) = \omega(g(n))$



<u>note</u>: $f(n) = \omega(g(n)) \Rightarrow f(n) = \Omega(g(n))$

Transitivity of: $O, \Omega, \Theta, o, \omega$

all satisfy transitivity.

<u>Exercise</u> show this.

say O:

show: $f(n) = O(g(n))$ and $g(n) = O(h(n))$

implies $f(n) = O(h(n))$.

hint:

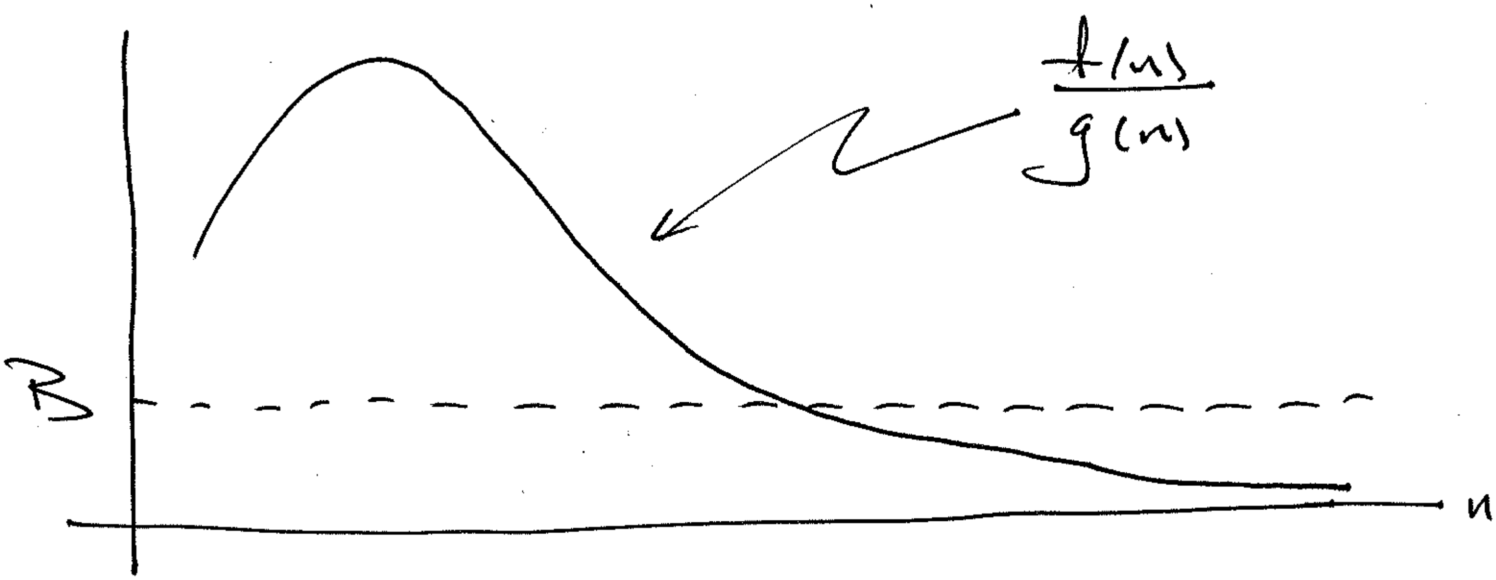
$$\frac{f(n)}{g(n)} \le B_1 \quad \text{and} \quad \frac{g(n)}{h(n)} \le B_2$$

implies

$$\frac{f(n)}{h(n)} = \frac{f(n)}{g(n)} \cdot \frac{g(n)}{h(n)} \le B_1 \cdot B_2$$

Also

$$f(n) = o(g(n)) \Longrightarrow f(n) \neq \Omega(g(n))$$



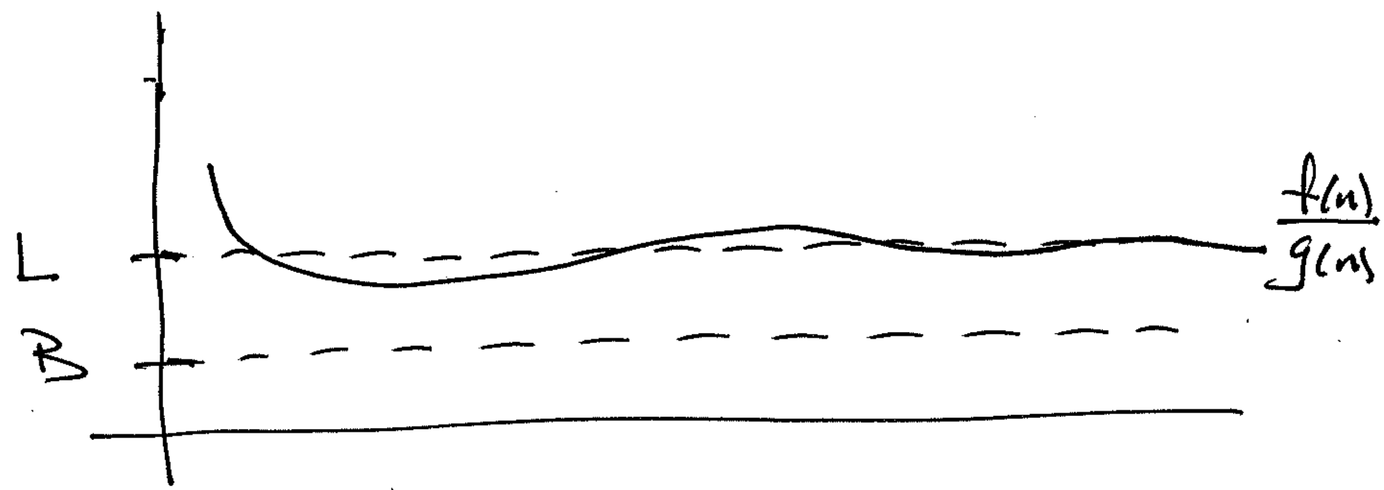
Similarly

$$f(n) = \omega(g(n)) \Longrightarrow f(n) \neq O(g(n)).$$

Exercise 5c:

Let $0 < L \le \infty$ and $L = \lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right)$.

Then $f(n) = \Omega(g(n))$.



exercise (1)

$$n^\alpha = \begin{cases} o(n^\beta) & \text{if } \alpha<\beta \\ \Theta(n^\beta) & \alpha=\beta \\ \omega(n^\beta) & \alpha>\beta \end{cases}$$

$$\frac{n^\alpha}{n^\beta} = n^{\alpha-\beta} \longrightarrow \begin{cases} 0 & \alpha<\beta \\ 1 & \alpha=\beta \\ \infty & \alpha>\beta \end{cases}$$

exercise (2)

$$a^n = \begin{cases} o(b^n) & \text{if } a<b \\ \Theta(b^n) & a=b \\ \omega(b^n) & a>b \end{cases}$$

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n \xrightarrow[n\to\infty]{\text{as}} \begin{cases} 0 & a<b \\ 1 & a=b \\ \infty & a>b \end{cases}$$

exercise (4)

$$\log_a(n) = \frac{\log_b(n)}{\log_b(a)}$$

$$\therefore \quad \log_b(n) = \underbrace{\log_b(a)}_{\text{const}} \cdot \log_a(n)$$

<u>hierarchy</u>:

$\vdots$

exponentials $\updownarrow$ a^n $(a>1)$

Polynomials $\updownarrow$ n^α $(\alpha>0)$

logs $\log_b(n)$

all same

$\vdots$

Exercise (6)

$$f(n) + o(f(n)) = \Theta(f(n))$$

why? let $h(n) = o(f(n))$, i.e.

$\frac{h(n)}{f(n)} \longrightarrow 0$ as $n \to \infty$. Then

$$\frac{f(n)+h(n)}{f(n)} = 1 + \underbrace{\frac{h(n)}{f(n)}}_{\downarrow \, 0} \longrightarrow 1 \quad \text{as } n \to \infty$$

$\therefore$ $f(n) + h(n) = \Theta(f(n))$.

Ex.

SelectionSort(A)

```
n = length[A]
for i = 1 to (n-1)
    imin = i
    for j = i+1 to n
        if A[i] < A[imin]
            imin = j
    A[i] <-> A[imin]      // swap
```

basic op.

$$\underbrace{A_1\, A_2 \,\cdots\, A_{i-1}}_{\text{sorted}}\ \underbrace{A_i\, A_{i+1} \,\cdots\, A_n}_{\text{unsorted section}}$$

sorted

(i-1) smallest elements

unsorted section

$$\# comp = (n-1) + (n-2) + \ldots + 3 + 2 + 1$$

$$= \frac{(n-1)(n-\cancel{1}+\cancel{1})}{2} = \frac{n(n-1)}{2}$$

$$= \frac{1}{2} n^2 - \frac{1}{2} n$$

$$= \Theta(n^2)$$