CSE 101-01 5-2-23

Recall:

- $f(n) = O(g(n))$ iff there exist positive B, n_0 s.t. for all $n \geq n_0$:

$$\frac{f(n)}{g(n)} \leq B$$

- $f(n) = \Omega(g(n))$ iff there exist pos. B, n_0, s.t. for all $n \geq n_0$

$$B \leq \frac{f(n)}{g(n)}$$

- $f(n) = \Theta(g(n))$ iff there exist pos. B_1, B_2, n_0 s.t. for all $n \geq n_0$:

$$B_1 \leq \frac{f(n)}{g(n)} \leq B_2$$

Analogy:

$$f(n) = O(g(n)) \longleftrightarrow x \leq y$$

$$f(n) = \Omega(g(n)) \longleftrightarrow x \geq y$$

$$f(n) = \Theta(g(n)) \longleftrightarrow x = y$$

$$f(n) = o(g(n)) \Longleftrightarrow x < y$$

$$f(n) = \omega(g(n)) \Longleftrightarrow x > y$$

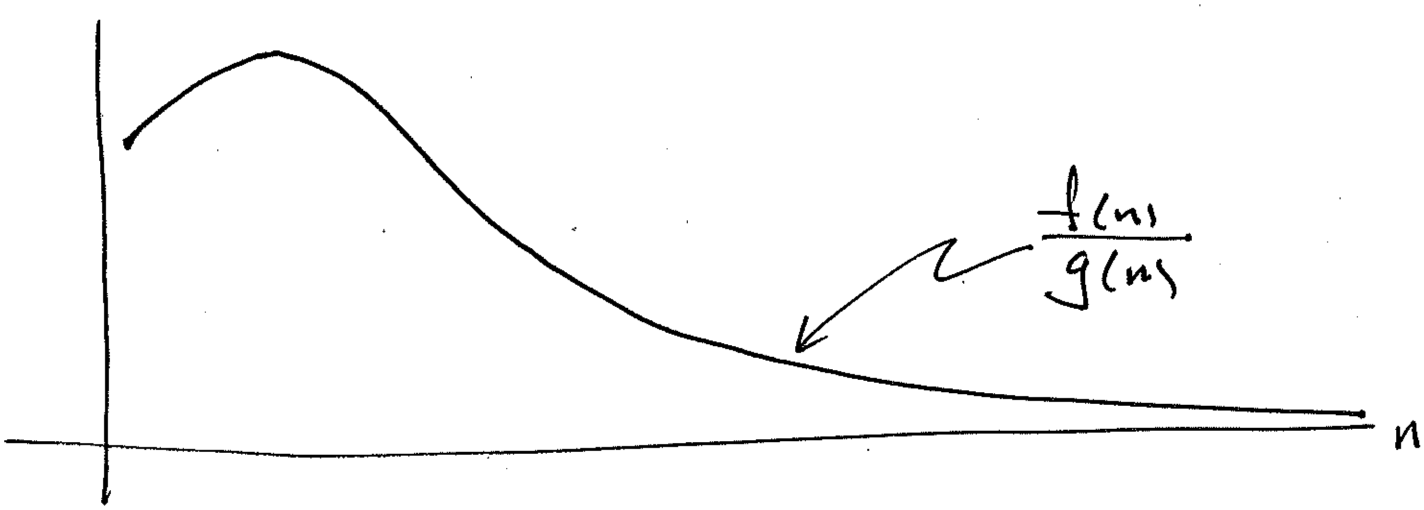
Defn

write $f(n) = o(g(n))$ iff

$$\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = 0$$

we say $g(n)$ is a strict asymptotic upper bound for $f(n)$.

Picture:



<u>Ex.</u> $\ln(n) = o(n)$ ✓

$$\lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right) = \lim_{n\to\infty}\left(\frac{1/n}{1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0 \quad ✓$$

<u>note</u> $\log_b(n) = \dfrac{\ln(n)}{\ln(b)}$

<u>Exercise</u>: show $\log_b(n) = o(n)$ for any $b > 1$

<u>Ex.</u> $n = o(n^2)$

$$\lim_{n\to\infty}\left(\frac{n}{n^2}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0 \quad ✓$$

<u>Exercise</u>: show for any $b > 1$, $k > 0$

$$\log_b(n) = o(n^k)$$

Ex. let $0 \le \alpha < \beta$. Then

$$n^{\alpha} = o(n^{\beta})$$

$$\lim_{n \to \infty} \left(\frac{n^{\alpha}}{n^{\beta}} \right) = \lim_{n \to \infty} n^{\alpha - \beta} = 0$$

Since $\alpha - \beta < 0$.

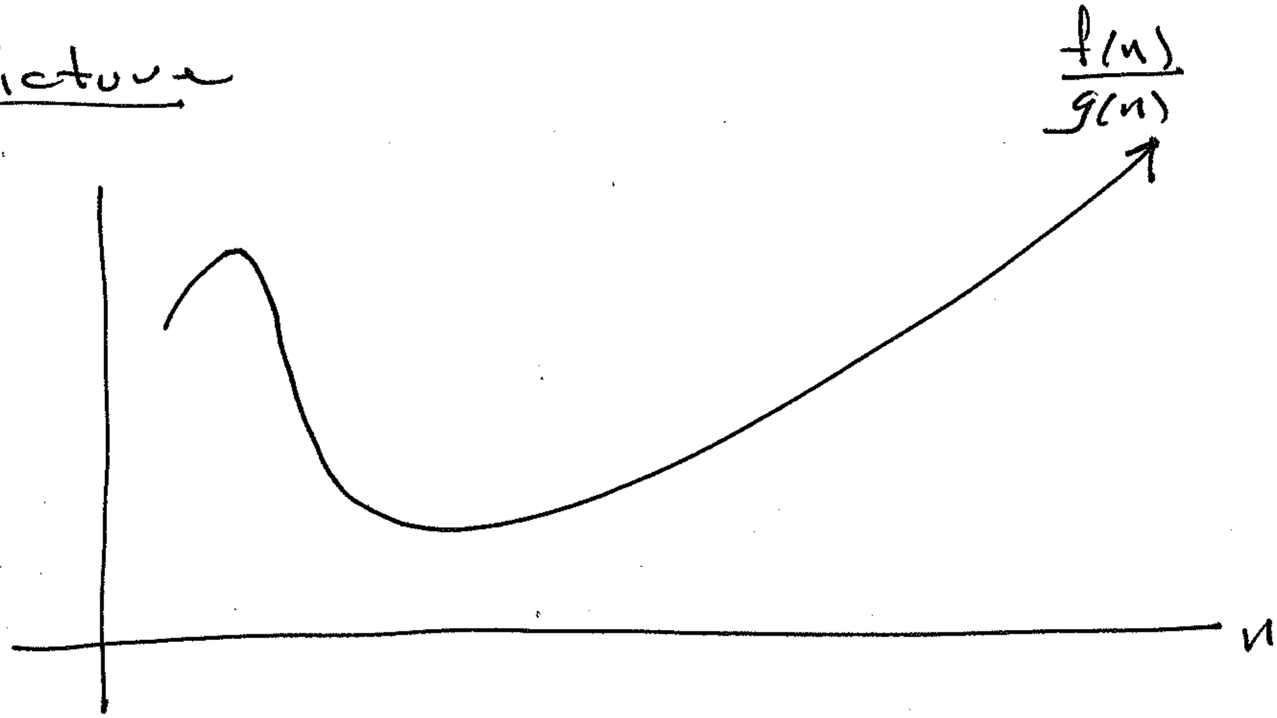
Defn

$f(n) = \omega(g(n))$ iff

$$\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = \infty$$

we say: $g(n)$ is a strict asymptotic lower bound for $f(n)$.

Picture



Note:

$$\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right)=0 \quad \text{iff} \quad \lim_{n\to\infty}\left(\frac{g(n)}{f(n)}\right)=\infty$$

Thus

$$f(n)=o(g(n)) \quad \text{iff} \quad g(n)=\omega(f(n))$$

Ex. $f(n) = b^n$, $g(n) = a^n$, $1 \le a < b$

Then $f(n) = \omega(g(n))$. why

$$\lim_{n \to \infty} \left(\frac{b^n}{a^n} \right) = \lim_{n \to \infty} \left(\frac{b}{a} \right)^n = \infty$$

Analogy: $O, \Omega, \Theta, o, \omega$ all satisfy transitivity. e.g.

$$f(n) = O(g(n)) \text{ and } g(n) = O(h(n)) \Rightarrow f(n) = O(h(n)).$$

Key fact:

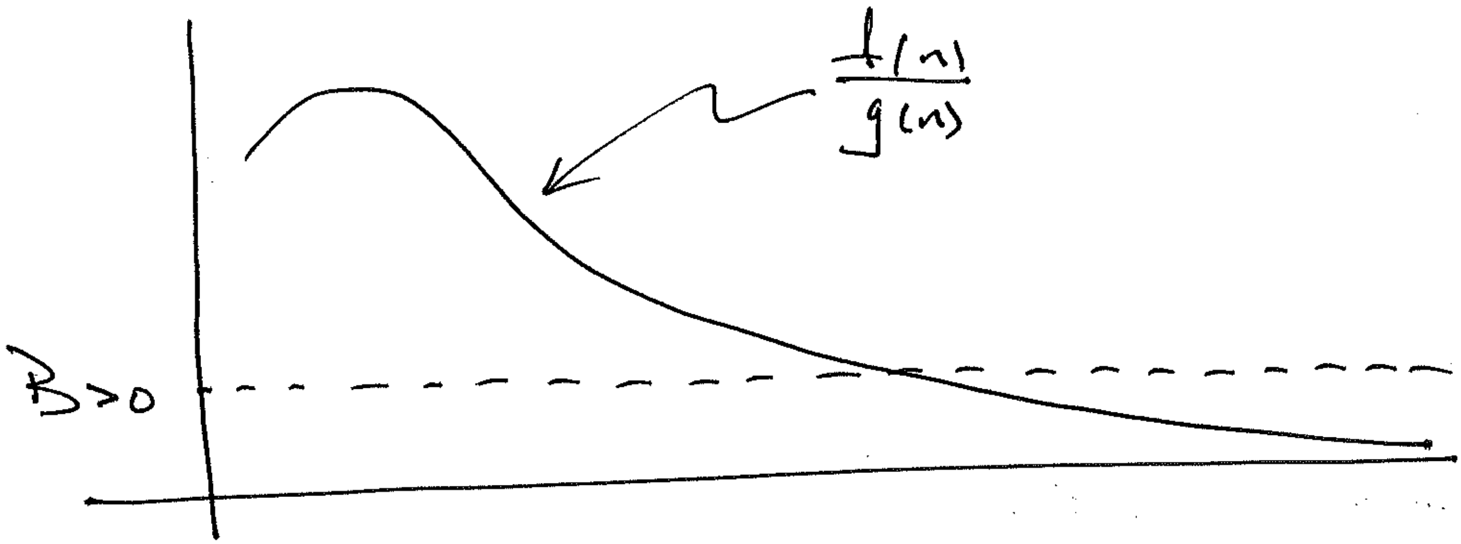
$$\frac{f(n)}{g(n)} \le B_1 \text{ and } \frac{g(n)}{h(n)} \le B_2 \Rightarrow$$

$$\frac{f(n)}{h(n)} = \frac{f(n)}{g(n)} \cdot \frac{g(n)}{h(n)} \le B_1 \cdot B_2$$

<u>Exercise</u>: carry out details

<u>Note</u>: $f(n) = o(g(n)) \Rightarrow f(n) = O(g(n))$

(but $\not\Leftarrow$.)



<u>Note</u>: $f(n) = o(g(n)) \Rightarrow f(n) \neq \Omega(g(n))$

also $f(n) = \omega(g(n)) \Rightarrow f(n) \neq O(g(n))$

More exercises

- $$\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right)=L, \text{ where } 0<L<\infty$$

then $f(n)=\Theta(g(n))$

Also
$$\begin{cases} 0<L\leq\infty \Rightarrow f(n)=\Omega(g(n)) \\ 0\leq L<\infty \Rightarrow f(n)=O(g(n)) \end{cases}$$

- $$n^\alpha=\begin{cases} o(n^\beta) & \text{if } \alpha<\beta \\ \Theta(n^\beta) & \text{if } \alpha=\beta \\ \omega(n^\beta) & \text{if } \alpha>\beta \end{cases}$$

$$\frac{n^\alpha}{n^\beta}=n^{\alpha-\beta}\longrightarrow\begin{cases} 0 & \alpha<\beta \\ 1 & \alpha=\beta \\ \infty & \alpha>\beta \end{cases}$$

- $$a^n = \begin{cases} o(b^n) & \text{if } a < b \\ \Theta(b^n) & \text{if } a = b \\ \omega(b^n) & \text{if } a > b \end{cases}$$

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n \longrightarrow \begin{cases} 0 & \text{if } a < b \\ 1 & \text{if } a = b \\ \infty & \text{if } a > b \end{cases}$$

- for any a, b with $a > 1$, $b > 1$

$$\log_b(n) = \Theta(\log_a(n)) \quad \checkmark$$

why $\log_a(n) = \dfrac{\log_b(n)}{\log_b(a)}$, i.e

$$\log_b(n) = \log_b(a) \cdot \log_a(n) = \text{Const} \cdot \log_a(n)$$

- for any fcn. $f(n)$

$$f(n) + o(f(n)) = \Theta(f(n))$$

why? let $h(n) = o(f(n))$. Then

$$\frac{f(n)+h(n)}{f(n)} = 1 + \underbrace{\frac{h(n)}{f(n)}}_{\downarrow \; 0} \longrightarrow 1$$

✓

Ex. Selection sort

SelectionSort(A)

```
n = length[A]
for i = 1 to (n-1)
    imin = i
    for j = i+1 to n
        if A[j] < A[imin]
            imin = j
    A[i] <-> A[imin]    // swap
```

basic operation

$$\underbrace{A_1 \cdots\cdots A_{i-1}}_{\substack{\text{sorted} \\ (i-1)\text{ smallest} \\ \text{elements}}} \; \underbrace{A_i \; A_{i+1} \cdots\cdots A_n}_{\text{unsorted}}$$

count #comparisons on input of size n:

$$\#\text{comp} = (n-1) + (n-2) + \cdots + 3 + 2 + 1$$

$$= \frac{(n-1)(n-\cancel{1}+\cancel{1})}{2} = \frac{n(n-1)}{2}$$

$$= \frac{1}{2}n^2 - \frac{1}{2}n$$

$$= \Theta(n^2)$$